Classification of selectors for sequences of dense sets of $C_p(X)$ ☆

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Abstract

For a Tychonoff space X, we denote by $C_p(X)$ the space of all real-valued continuous functions on X with the topology of pointwise convergence. In this paper we investigate different selectors for sequences of dense sets of $C_p(X)$. We give the characteristics of selection principles $S_1(\mathcal{P},\mathcal{Q})$, $S_{fin}(\mathcal{P},\mathcal{Q})$ and $U_{fin}(\mathcal{P},\mathcal{Q})$ for $\mathcal{P},\mathcal{Q}\in\{\mathcal{D},\,\mathcal{S},\,\mathcal{A}\}$, where

• $\mathcal{D}$ — the family of a dense subsets of $C_p(X)$;

• $\mathcal{S}$ — the family of a sequentially dense subsets of $C_p(X)$;

• $\mathcal{A}$ — the family of a 1-dense subsets of $C_p(X)$, through the selection principles of a space X.

Keywords:
$S_1(\mathcal{S},\mathcal{S})$, $U_{fin}(\mathcal{S},\mathcal{S})$, $S_1(\mathcal{D},\mathcal{S})$, $S_1(\mathcal{S},\mathcal{D})$, $S_{fin}(\mathcal{S},\mathcal{D})$, $S_1(\mathcal{D},\mathcal{D})$, $S_{fin}(\mathcal{D},\mathcal{D})$, $S_1(\mathcal{A},\mathcal{A})$, $U_{fin}(\mathcal{S},\mathcal{D})$, $S_1(\mathcal{S},\mathcal{A})$, $S_{fin}(\mathcal{A},\mathcal{A})$, function spaces, selection principles, C_p theory, Scheepers Diagram
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1. Introduction

Throughout this paper, all spaces are assumed to be Tychonoff. The set of positive integers is denoted by ω. Let $\mathbb{R}$ be the real line, we put $\mathbb{I}=[0,1]\subset\mathbb{R}$, and $\mathbb{Q}$ be the rational numbers. For a space X, we denote by $C_p(X)$ the space of all real-valued continuous functions on X with the topology of pointwise convergence. The symbol $\mathbf{0}$ stands for the constant function to 0.

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Basic open sets of $C_p(X)$ are of the form

$[x_1, ..., x_k, U_1, ..., U_k] = \{f \in C(X) : f(x_i) \in U_i,\ i = 1, ..., k\}$, where each $x_i \in X$ and each U_i is a non-empty open subset of $\mathbb{R}$. Sometimes we will write the basic neighborhood of the point f as $< f, A, \epsilon >$ where $< f, A, \epsilon > := \{g \in C(X) : |f(x) - g(x)| < \epsilon\ \forall x \in A\}$, A is a finite subset of X and $\epsilon > 0$.

In this paper, by cover we mean a nontrivial one, that is, $\mathcal{U}$ is a cover of X if $X = \cup\mathcal{U}$ and $X \notin \mathcal{U}$.

An open cover $\mathcal{U}$ of a space X is:

• an *ω-cover* if X does not belong to $\mathcal{U}$ and every finite subset of X is contained in a member of $\mathcal{U}$.

• a *γ-cover* if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$.

For a topological space X we denote:

• $\mathcal{O}$ — the family of open covers of X;
• Γ — the family of open γ-covers of X;
• Γ_{cl} — the family of clopen γ-covers of X;
• Ω — the family of open ω-covers of X;
• $\mathcal{D}$ — the family of a dense subsets of X;
• $\mathcal{S}$ — the family of a sequentially dense subsets of X.

Many topological properties are defined or characterized in terms of the following classical selection principles. Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$S_1(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $\{A_n : n \in \omega\}$ of elements of $\mathcal{A}$ there is a sequence $\{b_n\}_{n\in\omega}$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \omega\}$ is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $\{A_n : n \in \omega\}$ of elements of $\mathcal{A}$ there is a sequence $\{B_n\}_{n\in\omega}$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\omega} B_n \in \mathcal{B}$.

$U_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: whenever $\mathcal{U}_1, \mathcal{U}_2, ... \in \mathcal{A}$ and none contains a finite subcover, there are finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \omega$, such that $\{\bigcup \mathcal{F}_n : n \in \omega\} \in \mathcal{B}$.

The following prototype of many classical properties is called "$\mathcal{A}$ choose $\mathcal{B}$" in [39].

$\binom{\mathcal{A}}{\mathcal{B}}$: For each $\mathcal{U} \in \mathcal{A}$ there exists $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V} \in \mathcal{B}$.

Then $S_{fin}(\mathcal{A}, \mathcal{B})$ implies $\binom{\mathcal{A}}{\mathcal{B}}$.

Many equivalence hold among these properties, and the surviving ones appear in the following Diagram (where an arrow denote implication), to which no arrow can be added except perhaps from $U_{fin}(\Gamma,\Gamma)$ or $U_{fin}(\Gamma,\Omega)$ to $S_{fin}(\Gamma,\Omega)$ [16].

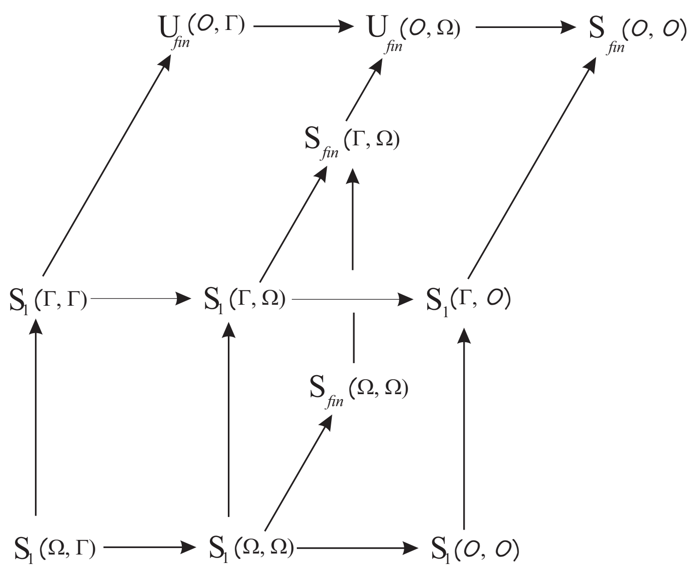

Fig. 1. The Scheepers Diagram.

The papers [16, 17, 32, 37, 41] have initiated the simultaneous consideration of these properties in the case where $\mathcal{A}$ and $\mathcal{B}$ are important families of open covers of a topological space X.

2. Main definitions and notation

Let X be a topological space, and $x \in X$. A subset A of X *converges* to x, $x = \lim A$, if A is infinite, $x \notin A$, and for each neighborhood U of x, $A \setminus U$ is finite. Consider the following collection:

- $\Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A\}$;
- $\Gamma_x = \{A \subseteq X : x = \lim A\}$.

We write $\Pi(\mathcal{A}_x, \mathcal{B}_x)$ without specifying x, we mean $(\forall x)\Pi(\mathcal{A}_x, \mathcal{B}_x)$.

- A space X has *countable fan tightness* (Arhangel'skii's countable fan tightness), if $X \models S_{fin}(\Omega_x, \Omega_x)$ [2].

• A space X has *countable strong fan tightness* (Sakai's countable strong fan tightness), if $X \models S_1(\Omega_x, \Omega_x)$ [28].

• A space X has *countable selectively sequentially fan tightness* (Arhangel'skii's property α_4), if $X \models S_{fin}(\Gamma_x, \Gamma_x)$ [1].

• A space X has *countable strong selectively sequentially fan tightness* (Arhangel'skii's property α_2), if $X \models S_1(\Gamma_x, \Gamma_x)$ [1].

• A space X has *strictly Fréchet-Urysohn at x,* if $X \models S_1(\Omega_x, \Gamma_x)$ [30].

• A space X has *almost strictly Fréchet-Urysohn at x*, if $X \models S_{fin}(\Omega_x, \Gamma_x)$.

• A space X has *the weak sequence selection property*, if $X \models S_1(\Gamma_x, \Omega_x)$ [33].

• A space X has *the sequence selection property*, if $X \models S_{fin}(\Gamma_x, \Omega_x)$.

The following implications hold

$$\begin{array}{ccccccc}
S_1(\Gamma_x, \Gamma_x) & \Rightarrow & S_{fin}(\Gamma_x, \Gamma_x) & \Rightarrow & S_1(\Gamma_x, \Omega_x) & \Rightarrow & S_{fin}(\Gamma_x, \Omega_x) \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
S_1(\Omega_x, \Gamma_x) & \Rightarrow & S_{fin}(\Omega_x, \Gamma_x) & \Rightarrow & S_1(\Omega_x, \Omega_x) & \Rightarrow & S_{fin}(\Omega_x, \Omega_x)
\end{array}$$

We write $\Pi(\mathcal{A}, \mathcal{B}_x)$ without specifying x, we mean $(\forall x)\Pi(\mathcal{A}, \mathcal{B}_x)$.

• A space X has *countable fan tightness with respect to dense subspaces,* if $X \models S_{fin}(\mathcal{D}, \Omega_x)$ ([6]).

• A space X has *countable strong fan tightness with respect to dense subspaces,* if $X \models S_1(\mathcal{D}, \Omega_x)$ ([6]).

• A space X has *almost strictly Fréchet-Urysohn at x with respect to dense subspaces*, if $X \models S_{fin}(\mathcal{D}, \Gamma_x)$.

• A space X has *strictly Fréchet-Urysohn at x with respect to dense subspaces*, if $X \models S_1(\mathcal{D}, \Gamma_x)$.

• A space X has *countable selectively sequentially fan tightness with respect to dense subspaces*, if $X \models S_{fin}(\mathcal{S}, \Gamma_x)$.

• A space X has *countable strong selectively sequentially fan tightness with respect to dense subspaces*, if $X \models S_1(\mathcal{S}, \Gamma_x)$.

• A space X has *the sequence selection property with respect to dense subspaces*, if $X \models S_{fin}(\mathcal{S}, \Omega_x)$.

• A space X has *the weak sequence selection property with respect to dense subspaces*, if $X \models S_1(\mathcal{S}, \Omega_x)$.

The following implications hold

$$\begin{array}{ccccccc}
S_1(\mathcal{S},\Gamma_x) & \Rightarrow & S_{fin}(\mathcal{S},\Gamma_x) & \Rightarrow & S_1(\mathcal{S},\Omega_x) & \Rightarrow & S_{fin}(\mathcal{S},\Omega_x)\\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow\\
S_1(\mathcal{D},\Gamma_x) & \Rightarrow & S_{fin}(\mathcal{D},\Gamma_x) & \Rightarrow & S_1(\mathcal{D},\Omega_x) & \Rightarrow & S_{fin}(\mathcal{D},\Omega_x)
\end{array}$$

• A space X is R-separable, if $X \models S_1(\mathcal{D},\mathcal{D})$ (Def. 47, [6]).
• A space X is M-separable (selective separability), if $X \models S_{fin}(\mathcal{D},\mathcal{D})$.
• A space X is selectively sequentially separable, if $X \models S_{fin}(\mathcal{S},\mathcal{S})$ (Def. 1.2, [7]).

The following implications hold

$$\begin{array}{ccccccc}
S_1(\mathcal{S},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{S},\mathcal{S}) & \Rightarrow & S_1(\mathcal{S},\mathcal{D}) & \Rightarrow & S_{fin}(\mathcal{S},\mathcal{D})\\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow\\
S_1(\mathcal{D},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{D},\mathcal{S}) & \Rightarrow & S_1(\mathcal{D},\mathcal{D}) & \Rightarrow & S_{fin}(\mathcal{D},\mathcal{D})
\end{array}$$

If X is a space and $A \subseteq X$, then the sequential closure of A, denoted by $[A]_{seq}$, is the set of all limits of sequences from A. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. If D is a countable sequentially dense subset of X then X call sequentially separable space.

Call X strongly sequentially dense in itself, if every dense subset of X is sequentially dense, and, X strongly sequentially separable, if X is separable and every countable dense subset of X is sequentially dense. Clearly, every strongly sequentially separable space is sequentially separable, and every sequentially separable space is separable.

We recall that a subset of X that is the complete preimage of zero for a certain function from $C(X)$ is called a zero-set. A subset $O \subseteq X$ is called a cozero-set (or functionally open) of X if $X \setminus O$ is a zero-set.

Recall that the i-weight $iw(X)$ of a space X is the smallest infinite cardinal number τ such that X can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than τ.

Theorem 2.1. *(Noble [21]) Let X be a space. A space $C_p(X)$ is separable if and only if $iw(X) = \aleph_0$.*

Definition 2.2. A space X has V**-property** ($X \models V$), if there exist a condensation (one-to-one continuous mapping) $f : X \mapsto Y$ from the space X on a separable metric space Y, such that $f(U)$ — F_σ-set of Y for any cozero-set U of X.

Theorem 2.3. *(Velichko [12]). Let X be a Tychonoff space. A space $C_p(X)$ is sequentially separable if and only if $X \models V$.*

Recall that the cardinal $\mathfrak{p}$ is the smallest cardinal so that there is a collection of $\mathfrak{p}$ many subsets of the natural numbers with the strong finite intersection property but no infinite pseudo-intersection. Note that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{c}$.

For $f, g \in \mathbb{N}^{\mathbb{N}}$, let $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many n. $\mathfrak{b}$ is the minimal cardinality of a $\leq^*$-unbounded subset of $\mathbb{N}^{\mathbb{N}}$. A set $B \subset [\mathbb{N}]^\infty$ is unbounded if the set of all increasing enumerations of elements of B is unbounded in $\mathbb{N}^{\mathbb{N}}$, with respect to $\leq^*$. It follows that $|B| \geq \mathfrak{b}$. A subset S of the real line is called a Q-set if each one of its subsets is a G_δ. The cardinal $\mathfrak{q}$ is the smallest cardinal so that for any $\kappa < \mathfrak{q}$ there is a Q-set of size κ. (See [10] for more on small cardinals including $\mathfrak{p}$).

3. $S_1(\mathcal{D}, \mathcal{D})$ — R-separable

In [28] (Lemma, Theorem 1), M. Sakai proved:

Theorem 3.1. (*Sakai*) *For each space X the following are equivalent.*

1. $C_p(X) \models S_1(\Omega_x, \Omega_x)$.
2. $X^n \models S_1(\mathcal{O}, \mathcal{O})$ *(X^n has Rothberger's property C'') for each $n \in \omega$.*
3. $X \models S_1(\Omega, \Omega)$.

In ([35], Theorem 13) M. Scheeper was proved the following result

Theorem 3.2. (*Scheeper*) *For each separable metric space X, the following are equivalent:*

1. $C_p(X) \models S_1(\mathcal{D}, \mathcal{D})$;
2. $X \models S_1(\Omega, \Omega)$.

By Theorem 57 in [6], [28] and Theorem 2.1, we have

Theorem 3.3. *For a space X, the following are equivalent:*

1. $C_p(X) \models S_1(\mathcal{D}, \mathcal{D})$;
2. $C_p(X) \models S_1(\Omega_x, \Omega_x)$, *and is separable;*
3. $C_p(X) \models S_1(\mathcal{D}, \Omega_x)$, *and is separable;*
4. $X \models S_1(\Omega, \Omega)$, *and* $iw(X) = \aleph_0$;
5. $X^n \models S_1(\mathcal{O}, \mathcal{O})$ *for each* $n \in \omega$, *and* $iw(X) = \aleph_0$.

Corollary 3.4. For a separable metrizable space X, the following are equivalent:

1. $C_p(X) \models S_1(\mathcal{D}, \mathcal{D})$ [R-separable];
2. $C_p(X) \models S_1(\Omega_x, \Omega_x)$ [countable strong fan tightness];
3. $C_p(X) \models S_1(\mathcal{D}, \Omega_x)$ [countable strong fan tightness with respect to dense subspaces];
4. $X \models S_1(\Omega, \Omega)$;
5. $X^n \models S_1(\mathcal{O}, \mathcal{O})$ for each $n \in \omega$ [X^n is Rothberger].

4. $S_{fin}(\mathcal{D}, \mathcal{D})$ — M-separable

In ([2], Theorem 2.2.2 in [4]) A.V. Arhangel'skii was proved the following result

Theorem 4.1. (*Arhangel'skii*) *For a space* X, *the following are equivalent:*

1. $C_p(X) \models S_{fin}(\Omega_x, \Omega_x)$;
2. $(\forall n \in \omega)\ X^n \models S_{fin}(\mathcal{O}, \mathcal{O})$.

It is known (see [16]) that $X \models S_{fin}(\Omega, \Omega)$ iff $(\forall n \in \omega)\ X^n \models S_{fin}(\mathcal{O}, \mathcal{O})$.

By Theorem 21 in [6] and Theorem 3.9 in [16], we have a next result.

Theorem 4.2. *For a space* X, *the following are equivalent:*

1. $C_p(X) \models S_{fin}(\mathcal{D}, \mathcal{D})$;
2. $X \models S_{fin}(\Omega, \Omega)$ *and* $iw(X) = \aleph_0$;
3. $(\forall n \in \omega)\ X^n \in S_{fin}(\mathcal{O}, \mathcal{O})$ *and* $iw(X) = \aleph_0$;
4. $C_p(X) \models S_{fin}(\Omega_x, \Omega_x)$ *and is separable;*
5. $C_p(X) \models S_{fin}(\mathcal{D}, \Omega_x)$ *and is separable.*

Corollary 4.3. For a separable metrizable space X, the following are equivalent:

1. $C_p(X) \models S_{fin}(\mathcal{D}, \mathcal{D})$ [M-separable];
2. $C_p(X) \models S_{fin}(\Omega_x, \Omega_x)$ [countable fan tightness];
3. $C_p(X) \models S_{fin}(\mathcal{D}, \Omega_x)$ [countable strong fan tightness with respect to dense subspaces];
4. $X \models S_{fin}(\Omega, \Omega)$;
5. $X^n \models S_{fin}(\mathcal{O}, \mathcal{O})$ for each $n \in \omega$ [X^n is Menger].

5. $S_1(\mathcal{D}, \mathcal{S})$

Pytkeev [26] and independently Gerlits [14], see also [4] and [20], proved

Theorem 5.1. *For a space X, the following statements are equivalent:*

1. *$C_p(X)$ is Fréchet-Urysohn;*
2. *$C_p(X)$ is sequential;*
3. *$C_p(X)$ is a k-space.*

Gerlits and Nagy [13] proved

Theorem 5.2. *(Gerlits, Nagy) For a space X, the following statements are equivalent:*

1. *$C_p(X) \models S_1(\Omega_x, \Gamma_x)$;*
2. *$C_p(X)$ is Fréchet-Urysohn;*
3. *$X \models S_1(\Omega, \Gamma)$;*
4. *$X \models \binom{\Omega}{\Gamma}$.*

Theorem 5.3. *For a space X, the following statements are equivalent:*

1. *$C_p(X)$ is strongly sequentially dense in itself;*
2. *$X \models S_1(\Omega, \Gamma)$.*

Proof. (1) $\Rightarrow$ (2). By Theorem 5.2, $S_1(\Omega, \Gamma) = \binom{\Omega}{\Gamma}$. Let $\mathcal{U} \in \Omega$ and P be a dense subset of $C_p(X)$. A set $\mathcal{D} := \{f \in C(X) : f \upharpoonright K = h$ for $h \in P$, and $f \upharpoonright (X \setminus U) = 1$ for a finite subset $K \subset U$ where $U \in \mathcal{U}\}$.

Since $\mathcal{U}$ is a ω-cover of X and P is a dense subset of $C_p(X)$, we claim that $\mathcal{D}$ is a dense subset of $C_p(X)$.

Fix $g \in C(X)$. Let K be a finite subset of X, $\epsilon > 0$ and $W =< g, K, \epsilon >$ be a base neighborhood of g, then there is $U \in \mathcal{U}$ such that $K \subset U$ and $h \in W$ for some $h \in P$. Since $f \upharpoonright K = h \upharpoonright K$ for some $f \in \mathcal{D}$, then

$f \in W$. By (1), $\mathcal{D}$ is a sequentially dense subset of $C_p(X)$. Then there exists a sequence $\{f_i\}_{i\in\omega}$ such that for each i, $f_i \in \mathcal{D}$, and $\{f_i\}_{i\in\omega}$ converge to $\mathbf{0}$. By definition of f_i, $f_i \upharpoonright K_i = h_i$ for some finite set K_i and $h_i \in P$, and $f_i \upharpoonright (X \setminus U_i) = 1$ for some $U_i \in \mathcal{U}$.

Consider a sequence $\{U_i\}_{i\in\omega}$.

(a). $U_i \in \mathcal{U}$.

(b). $\{U_i : i \in \omega\}$ is a γ-cover of X.

Let K be a finite subset of X and $W = [K, (-1, 1)]$ be a base neighborhood of $\mathbf{0}$, then there is $i' \in \omega$ such that $f_i \in W$ for each $i > i'$. It follows that $K \subset U_i$ for each $i > i'$. We thus get that $X \models \binom{\Omega}{\Gamma}$, and, hence, $X \models S_1(\Omega, \Gamma)$.

(2) $\Rightarrow$ (1). By Theorem 5.2, $C_p(X)$ is Fréchet-Urysohn, and, hence, $C_p(X)$ is strongly sequentially dense in itself. $\square$

Corollary 5.4. For a space X, the following statements are equivalent:

1. $C_p(X) \models S_1(\Omega_x, \Gamma_x)$;
2. $C_p(X)$ is Fréchet-Urysohn;
3. $X \models S_1(\Omega, \Gamma)$;
4. $X \models \binom{\Omega}{\Gamma}$;
5. $C_p(X)$ is sequential;
6. $C_p(X)$ is a k-space;
7. $C_p(X)$ is strongly sequentially dense in itself.

Note that $S_1(\Omega, \Gamma) = S_{fin}(\Omega, \Gamma)$ (see [16]).

Theorem 5.5. *For a space X, the following statements are equivalent:*

1. $X \models S_{fin}(\Omega, \Gamma)$;
2. $C_p(X) \models S_{fin}(\Omega_x, \Gamma_x)$.

Proof. By Theorem 5.2, it suffices to prove (2) $\Rightarrow$ (1).

(2) $\Rightarrow$ (1). Let $\{\mathcal{U}_n\}_{n\in\omega}$ be a sequence of open ω-covers of X. We set $A_n = \{f \in C(X) : f \upharpoonright (X \setminus U) = 0 \text{ for } U \in \mathcal{U}_n\}$. It is not difficult to see that each A_n is dense in $C(X)$ since each $\mathcal{U}_n$ is an ω-cover of X and X is Tychonoff. Let f be the constant function to 1. By the assumption there exist $\{f_n^i : i = 1, ..., k(n)\} \subset A_n$ such that $\bigcup_{n\in\omega}\{f_n^i\}_{i=1}^{k(n)}$ converge to f. Consider subsequence $\{f_n^1\}_{n\in\omega} \subset \bigcup_{n\in\omega}\{f_n^i\}_{i=1}^{k(n)}$. Note that $\{f_n^1\}_{n\in\omega}$ also converge to f.

For each f_n^1 we take $U_n \in \mathcal{U}_n$ such that $f_n^1 \upharpoonright (X \setminus U_n) = 0$.

Set $\mathcal{U} = \{U_n : n \in \omega\}$. For each finite subset $\{x_1, ..., x_k\}$ of X we consider the basic open neighborhood of f $[x_1, ..., x_k; W, ..., W]$, where $W = (0, 2)$.

Note that there is $n' \in \omega$ such that $[x_1, ..., x_k; W, ..., W]$ contains f_n^1 for $n > n'$. This means $\{x_1, ..., x_k\} \subset U_n$ for $n > n'$. Consequently $\mathcal{U}$ is an γ-cover of X.

□

Theorem 5.6. *For a space X, the following statements are equivalent:*

1. $C_p(X) \models S_1(\mathcal{D}, \mathcal{S})$*;*
2. $C_p(X)$ *is strongly sequentially dense in itself and is separable;*
3. $X \models S_1(\Omega, \Gamma)$ *and* $iw(X) = \aleph_0$*;*
4. $C_p(X) \models S_1(\Omega_x, \Gamma_x)$ *and is separable;*
5. $C_p(X) \models S_1(\mathcal{D}, \Gamma_x)$ *and is separable;*
6. $C_p(X) \models S_{fin}(\mathcal{D}, \mathcal{S})$*;*
7. $X \models S_{fin}(\Omega, \Gamma)$ *and* $iw(X) = \aleph_0$*;*
8. $C_p(X) \models S_{fin}(\Omega_x, \Gamma_x)$ *and is separable;*
9. $C_p(X) \models S_{fin}(\mathcal{D}, \Gamma_x)$ *and is separable.*

Proof. (1) $\Rightarrow$ (2). Let D be a dense subset of $C_p(X)$. By $S_1(\mathcal{D}, \mathcal{S})$, for sequence $\{D_i : D_i = D \text{ and } i \in \omega\}$ there is a sequence $(d_i : i \in \omega)$ such that for each i, $d_i \in D_i$, and $\{d_i : i \in \omega\}$ is a countable sequentially dense subset of $C_p(X)$. It follows that D is a sequentially dense subset of $C_p(X)$.

(2) $\Rightarrow$ (1). Let $\{D_i\}_{i \in \omega}$ be a sequence of dense subsets of $C_p(X)$. Since $X \models S_1(\Omega, \Gamma)$, then, $X \models S_1(\Omega, \Omega)$ and, by Theorem 3.3, $C_p(X) \models S_1(\mathcal{D}, \mathcal{D})$. Then there is a sequence $\{d_i\}_{i \in \omega}$ such that for each i, $d_i \in D_i$, and $\{d_i : i \in \omega\}$ is a countable dense subset of $C_p(X)$. By (2), $\{d_i : i \in \omega\}$ is a countable sequentially dense subset of $C_p(X)$, i.e. $\{d_i : i \in \omega\} \in \mathcal{S}$.

(2) $\Rightarrow$ (3). By Theorem 5.3 and Theorem 2.1.

(3) $\Leftrightarrow$ (4). By Theorem 5.2.

(4) $\Rightarrow$ (5) is immediate.

(5) $\Rightarrow$ (2). Let $D \in \mathcal{D}$, $f \in C(X)$ and $\{D_n\}_{n \in \omega}$ such that $D_n = D$ for each $n \in \omega$. By (5), there is a sequence $\{f_n\}_{n \in \omega}$ such that for each n, $f_n \in D_n$, and $\{f_n\}_{n \in \omega}$ converge to f. It follows that D is a sequentially dense subset of $C_p(X)$.

(7) $\Leftrightarrow$ (8). By Theorem 5.5 and Theorem 2.1.

(8) $\Rightarrow$ (9) is immediate.

(1) $\Rightarrow$ (6) is immediate.
(3) $\Rightarrow$ (7) is immediate.
(9) $\Rightarrow$ (2) is proved similarly the implication (5) $\Rightarrow$ (2).
(6) $\Rightarrow$ (2) is proved similarly the implication (1) $\Rightarrow$ (2).

$\square$

Corollary 5.7. For a separable metrizable space X, the following are equivalent:

1. $C_p(X) \models S_1(\mathcal{D}, \mathcal{S})$;
2. $C_p(X)$ is strongly sequentially dense in itself;
3. $C_p(X)$ is strongly sequentially separable;
4. $C_p(X) \models S_1(\Omega_x, \Gamma_x)$;
5. $C_p(X) \models S_1(\mathcal{D}, \Gamma_x)$;
6. $X \models S_1(\Omega, \Gamma)$;
7. $C_p(X) \models S_{fin}(\mathcal{D}, \mathcal{S})$;
8. $C_p(X) \models S_{fin}(\Omega_x, \Gamma_x)$;
9. $C_p(X) \models S_{fin}(\mathcal{D}, \Gamma_x)$;
10. $X \models S_{fin}(\Omega, \Gamma)$.

6. $S_1(\mathcal{S}, \mathcal{D})$

In [29] (Theorem 2.3), M. Sakai proved:

Theorem 6.1. *(Sakai) For a space X, the following statements are equivalent:*

1. $C_p(X) \models S_1(\Gamma_x, \Omega_x)$ *(the weak sequence selection property);*
2. $X \models S_1(\Gamma_{cl}, \Omega_{cl})$ *and is strongly zero-dimensional.*

Definition 6.2. (Sakai) An γ-cover $\mathcal{U}$ of co-zero sets of X is γ_F**-shrinkable** if there exists a γ-cover $\{F(U) : U \in \mathcal{U}\}$ of zero-sets of X with $F(U) \subset U$ for every $U \in \mathcal{U}$.

For a topological space X we denote:
• Γ_F — the family of γ_F-shrinkable γ-covers of X.

Proposition 6.3. *For a strongly zero-dimensional space X, the following statements are equivalent:*

1. $X \models S_1(\Gamma_F, \Omega)$;
2. $X \models S_1(\Gamma_{cl}, \Omega_{cl})$.

Proposition 6.4. *For a space X, the following statements are equivalent:*

1. $C_p(X) \models S_1(\Gamma_x, \Omega_x)$;
2. $X \models S_1(\Gamma_F, \Omega)$.

Proof. (1) $\Rightarrow$ (2). By Theorem 6.1 and Proposition 6.3.

(2) $\Rightarrow$ (1). Let $X \models S_1(\Gamma_F, \Omega)$ and $\{A_i\}_{i\in\omega}$ such that $A_i \in \Gamma_{\mathbf{0}}$ for each $i \in \omega$. Consider $\mathcal{U}_i = \{f^{-1}(-\frac{1}{i}, \frac{1}{i}) : f \in A_i\}$ for each $i \in \omega$. Without loss of generality we can assume that there is i' that a set $U \neq X$ for any $i > i'$ and $U \in \mathcal{U}_i$. Otherwise there is sequence $\{f_{i_k}\}_{k\in\omega}$ such that $\{f_{i_k}\}_{k\in\omega}$ uniform converge to $\mathbf{0}$ and $\{f_{i_k} : k \in \omega\} \in \Omega_{\mathbf{0}}$.

Note that $\mathcal{F}_i = \{f^{-1}[-\frac{1}{i+1}, \frac{1}{i+1}] : f \in A_i\}$ is γ-cover of zero-sets of X. It follows that $\mathcal{U}_i \in \Gamma_F$ for each $i \in \omega$. By $X \models S_1(\Gamma_F, \Omega)$, there is a set $\{U_i : i \in \omega\}$ such that for each i, $U_i \in \mathcal{U}_i$, and $\{U_i : i \in \omega\}$ is an element of Ω.

We claim that $\mathbf{0} \in \overline{\{f_i : i \in \omega\}}$. Let $W =< \mathbf{0}\ , K, \epsilon >$ be a base neighborhood of $\mathbf{0}$ where $\epsilon > 0$ and K is a finite subset of X, then there are $i_0 \in \omega$ such that $\frac{1}{i_0} < \epsilon$ and $U_{i_0} \supset K$. It follows that $f_{i_0} \in W$ and, hence, $\mathbf{0} \in \overline{\{f_i : i \in \omega\}}$ and $C_p(X) \models S_1(\Gamma_x, \Omega_x)$.

By Theorem 6.1, we have that X is strongly zero-dimensional.

$\square$

Theorem 6.5. *For a space X, the following statements are equivalent:*

1. $C_p(X) \models S_1(\mathcal{S}, \mathcal{D})$ *and is sequentially separable;*
2. $X \models S_1(\Gamma_F, \Omega)$, $X \models V$;
3. $X \models S_1(\Gamma_{cl}, \Omega_{cl})$, $X \models V$ *and is strongly zero-dimensional;*
4. $C_p(X) \models S_1(\Gamma_x, \Omega_x)$ *and is sequentially separable;*
5. $C_p(X) \models S_1(\mathcal{S}, \Omega_x)$ *and is sequentially separable.*

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{V}_i\} \subset \Gamma_F$ and $\mathcal{S} = \{h_m : m \in \omega\}$ be a countable sequentially dense subset of $C_p(X)$. For each $i \in \omega$ we consider a countable sequentially dense subset $\mathcal{S}_i$ of $C_p(X)$ and $\mathcal{U}_i = \{U_i^m\}_{m\in\omega}$ where $\mathcal{U}_i \subset \mathcal{V}_i$ and $\mathcal{S}_i = \{f_i^m \in C(X) : f_i^m \upharpoonright F(U_i^m) = h_m$ and $f_i^m \upharpoonright (X \setminus U_i^m) = 1$ for $m \in \omega\}$. Note that $\mathcal{U}_i \in \Gamma_F$ for each $i \in \omega$.

Since $\mathcal{F}_i = \{F(U_i^m) : m \in \omega\}$ is infinite, it is a γ-cover of zero subsets of X. Since $\mathcal{S}$ is a countable sequentially dense subset of $C_p(X)$, we have that $\mathcal{S}_i$ is a countable sequentially dense subset of $C_p(X)$ for each $i \in \omega$. Let $h \in C(X)$, there is a sequence $\{h_{m_s} : s \in \omega\} \subset \mathcal{S}$ such that $\{h_{m_s}\}_{s\in\omega}$ converge to h. Let K be a finite subset of X, $\epsilon > 0$ and $W =< h, K, \epsilon >$ be a base neighborhood of h, then there is a number m_0 such that $K \subset F(U_i^m)$ for $m > m_0$ and $h_{m_s} \in W$ for $m_s > m_0$. Since $f_i^{m_s} \upharpoonright K = h_{m_s} \upharpoonright K$ for each $m_s > m_0$, $f_i^{m_s} \in W$ for each $m_s > m_0$. It follows that a sequence $\{f_i^{m_s}\}_{s\in\omega}$ converge to h.

By $C(X) \in S_1(\mathcal{S}, \mathcal{D})$, there is a sequence $\{f_i^{m(i)} : i \in \omega\}$ such that for each i, $f_i^{m(i)} \in \mathcal{S}_i$, and $\{f_i^{m(i)} : i \in \omega\}$ is an element of $\mathcal{D}$.

Consider a set $\{U_i^{m(i)} : i \in \omega\}$.

(a). $U_i^{m(i)} \in \mathcal{U}_i$.

(b). $\{U_i^{m(i)} : i \in \omega\}$ is a ω-cover of X.

Let K be a finite subset of X and $U =< \mathbf{0}\ , K, (-1, 1) >$ be a base neighborhood of $\mathbf{0}$, then there is $f_{i_{j_0}}^{m(i)_{j_0}} \in U$ for some $j_0 \in \omega$. It follows that $K \subset U_{i_{j_0}}^{m(i)_{j_0}}$. We thus get $X \models S_1(\Gamma_F, \Omega)$.

$(2) \Rightarrow (4)$. By Proposition 6.4 and Theorem 2.3.

$(2) \Leftrightarrow (3)$. Clearly, that $\Gamma_{cl} \subset \Gamma_F$. For a strongly zero-dimensional X, if $\mathcal{U} \in \Gamma_F$, then there is $\mathcal{W} \in \Gamma_{cl}$ such that $F(U) \subset W \subset U$ for every $U \in \mathcal{U}$ and $W \in \mathcal{W}$.

$(3) \Leftrightarrow (4)$. By Theorem 2.3 and Theorem 6.1.

$(4) \Rightarrow (5)$ is immediate.

$(5) \Rightarrow (1)$. Suppose that $C_p(X)$ is sequentially separable and $C_p(X) \models S_1(\mathcal{S}, \Omega_x)$.

Let $D = \{d_n : n \in \omega\}$ be a dense subspace of $C_p(X)$. Given a sequence of sequentially dense subspace of $C_p(X)$, enumerate it as $\{S_{n,m} : n, m \in \omega\}$. For each $n \in \omega$, pick $d_{n,m} \in S_{n,m}$ so that $d_n \in \overline{\{d_{n,m} : m \in \omega\}}$. Then $\{d_{n,m} : m, n \in \omega\}$ is dense in $C_p(X)$.

$\square$

Corollary 6.6. For a separable metrizable space X, the following statements are equivalent:

1. $C_p(X) \models S_1(\mathcal{S}, \mathcal{D})$;
2. $X \models S_1(\Gamma_F, \Omega)$;
3. $X \models S_1(\Gamma_{cl}, \Omega_{cl})$, and is strongly zero-dimensional;

4. $C_p(X) \models S_1(\Gamma_x, \Omega_x)$;
5. $C_p(X) \models S_1(\mathcal{S}, \Omega_x)$.

7. $S_{fin}(\mathcal{S}, \mathcal{D})$

Theorem 7.1. *For a space X, the following statements are equivalent:*

1. $C_p(X) \models S_{fin}(\Gamma_x, \Omega_x)$;
2. $X \models S_{fin}(\Gamma_F, \Omega)$.

Proof. (1) $\Rightarrow$ (2) Let $\{\mathcal{V}_n\}_{n\in\omega}$ be a sequence γ_F-shrinkable γ-covers of X. Let $\mathcal{U}_n = \{U_{n,m} : m \in \omega\} \subset \mathcal{V}_n$ for each $n \in \omega$. Note that $\mathcal{U}_n \in \Gamma_F$ for each $n \in \omega$.

For $n, m \in \omega$, let $f_{n,m} : X \mapsto [0, 1]$ be the continuous function satisfying $F(U_{n,m}) = f^{-1}_{n,m}(0)$ and $X \setminus U_{n,m} = f^{-1}_{n,m}(1)$. For each $n \in \omega$, $\{F(U_{n,m}) : m \in \omega\}$ is a γ-cover of X, it follows that $\{f_{n,m}\}_{m\in\omega}$ is a sequence converging pointwise to $\mathbf{0}$. By $C_p(X) \models S_{fin}(\Gamma_x, \Omega_x)$, there is a sequence $\{F_n = \{f_{n,m_1}, f_{n,m_2}, ..., f_{n,m_{k_n}}\}\}_{n\in\omega}$ such that $F_n \subset \{f_{n,m}\}_{m\in\omega}$ for each $n \in \omega$ and $\bigcup_{n\in\omega} F_n \in \Omega_0$. Then $\bigcup_{n\in\omega}\{U_{n,m_1}, U_{n,m_2}, ..., U_{n,m_{k_n}}\}$ is an ω-cover of X.

(2) $\Rightarrow$ (1). For each $n \in \omega$, let $A_n \in \Gamma_{\mathbf{0}}$.

For $n \in \omega$ and $f \in A_n$, let $Z_{n,f} = \{x \in X : |f(x)| \leq \frac{1}{2^{n+1}}\}$, $U_{n,f} = \{x \in X : |f(x)| < \frac{1}{2^n}\}$. For each $n \in \omega$, we put $\mathcal{U}_n = \{U_{n,f} : f \in A_n\}$. If the set $\{n \in \omega : X \in \mathcal{U}_n\}$ is infinite, $X = U_{n_1,f_1} = U_{n_2,f_2} = ...$ for some sequences $\{n_j\}_{j\in\omega}$ and $f_i \in A_{n_i}$, where $\{n_j\}_{j\in\omega}$ is strictly increasing. This means that $\{f_i\}_{i\in\omega}$ is a sequence converging uniformly to $\mathbf{0}$. If the set $\{n \in \omega : X \in \mathcal{U}_n\}$ is finite, by removing such finitely many n's we assume $U_{n,f} \neq X$ for $n \in \omega$ and $f \in A_n$.

Note that each $\mathcal{U}_n$ is a γ_F-shrinkable γ-covers of X. By $X \models S_{fin}(\Gamma_F, \Omega)$, there is a sequence $\{U_{n,f_{n,1}}, ..., U_{n,f_{n,k(n)}}\}_{n\in\omega}$ such that $U_{n,f_{n,i}} \in \mathcal{U}_n$ for each $n \in \omega$, $i \in \overline{1, k(n)}$ and $\bigcup_{n\in\omega}\{U_{n,f_{n,1}}, U_{n,f_{n,2}}, ..., U_{n,f_{n,k(n)}}\}$ is an ω-cover of X. We claim a sequence $\{F_n = \{f_{n,1}, f_{n,2}, ..., f_{n,k(n)}\}\}_{n\in\omega}$ such that $F_n \subset A_n$ for each $n \in \omega$ and $\bigcup_{n\in\omega} F_n \in \Omega_0$.

Let K be a finite subset of X and let ϵ a positive real number.

Because of $U_{n,f_{n,i}} \neq X$ for $n \in \omega$ and $i \in \overline{1, k(n)}$, there are $n' \in \omega$ and $i' \in \overline{1, k(n')}$ such that $K \subset U_{n',f_{n',i'}}$ and $\frac{1}{2^{n'}} < \epsilon$. Then $|f_{n',i'}(x)| < \epsilon$ for any $x \in K$. Thus $C_p(X) \models S_{fin}(\Gamma_x, \Omega_x)$.

$\square$

Theorem 7.2. *For a space X, the following statements are equivalent:*

1. $C_p(X) \models S_{fin}(\mathcal{S}, \mathcal{D})$ *and is sequentially separable;*
2. $X \models S_{fin}(\Gamma_F, \Omega)$, $X \models V$*;*
3. $C_p(X) \models S_{fin}(\Gamma_x, \Omega_x)$ *and is sequentially separable;*
4. $C_p(X) \models S_{fin}(\mathcal{S}, \Omega_x)$ *and is sequentially separable.*

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{V}_i\} \subset \Gamma_F$ and $\mathcal{S} = \{h_m : m \in \omega\}$ be a countable sequentially dense subset of $C_p(X)$. For each $i \in \omega$ we consider a countable sequentially dense subset $\mathcal{S}_i$ of $C_p(X)$ and $\mathcal{U}_i = \{U_i^m : m \in \omega\}$ where $\mathcal{U}_i \subset \mathcal{V}_i$ and

$\mathcal{S}_i = \{f_i^m \in C(X) : f_i^m \upharpoonright F(U_i^m) = h_m$ and $f_i^m \upharpoonright (X \setminus U_i^m) = 1$ for $m \in \omega\}$. Note that $\mathcal{U}_i \in \Gamma_F$ for each $i \in \omega$.

Since $\mathcal{F}_i = \{F(U_i^m) : m \in \omega\}$ is infinite, it is a γ-cover of zero subsets of X. Since $\mathcal{S}$ is a countable sequentially dense subset of $C_p(X)$, we have that $\mathcal{S}_i$ is a countable sequentially dense subset of $C_p(X)$ for each $i \in \omega$. Let $h \in C(X)$, there is a sequence $\{h_{m_s}\}_{s\in\omega} \subset \mathcal{S}$ such that $\{h_{m_s}\}_{s\in\omega}$ converge to h. Let K be a finite subset of X, $\epsilon > 0$ and $W =< h, K, \epsilon >$ be a base neighborhood of h, then there is a number m_0 such that $K \subset F(U_i^m)$ for $m > m_0$ and $h_{m_s} \in W$ for $m_s > m_0$. Since $f_i^{m_s} \upharpoonright K = h_{m_s} \upharpoonright K$ for each $m_s > m_0$, $f_i^{m_s} \in W$ for each $m_s > m_0$. It follows that a sequence $\{f_i^{m_s}\}_{s\in\omega}$ converge to h.

By $C(X) \in S_{fin}(\mathcal{S}, \mathcal{D})$, there is a sequence $\{F_i = \{f_{i,m_1}, f_{i,m_2}, ..., f_{i,m_{k_i}}\}\}_{i\in\omega}$ such that $F_i \subset \mathcal{S}_i$ for each $i \in \omega$ and $\bigcup_{i\in\omega} F_i \in \mathcal{D}$. Then $\bigcup_{i\in\omega}\{U_{i,m_1}, U_{i,m_2}, ..., U_{i,m_{k_i}}\}$ is an ω-cover of X.

Let K be a finite subset of X and $U =< \mathbf{0}\ , K, (-1, 1) >$ be a base neighborhood of $\mathbf{0}$, then there is $f_{i',m(i')} \in \bigcup_{i\in\omega} F_i$ for some $i' \in \omega$ such that $f_{i',m(i')} \in U$. It follows that $K \subset U_{i'}^{m(i')}$. We thus get $X \models S_1(\Gamma_F, \Omega)$.

(2) $\Rightarrow$ (3). By Theorem 7.1 and Theorem 2.3.

(3) $\Rightarrow$ (4) is immediate.

(4) $\Rightarrow$ (1). Suppose that $C_p(X)$ is sequentially separable and $C_p(X) \models S_{fin}(\mathcal{S}, \Omega_x)$.

Let $D = \{d_n : n \in \omega\}$ be a dense subspace of $C_p(X)$. Given a sequence of sequentially dense subspace of $C_p(X)$, enumerate it as $\{S_{n,m} : n, m \in \omega\}$. For each $n \in \omega$, pick $D_{n,m} = \{d_{n,m(1)}, ..., d_{n,m(n)}\} \subset S_{n,m}$ so that $d_n \in \overline{\bigcup_{m\in\omega} D_{n,m}}$. Then $\bigcup_{n,m\in\omega} D_{n,m}$ is dense in $C_p(X)$.

$\square$

Corollary 7.3. For a separable metrizable space X, the following statements are equivalent:

1. $C_p(X) \models S_{fin}(\mathcal{S}, \mathcal{D})$;
2. $X \models S_{fin}(\Gamma_F, \Omega)$;
3. $C_p(X) \models S_{fin}(\Gamma_x, \Omega_x)$;
4. $C_p(X) \models S_{fin}(\mathcal{S}, \Omega_x)$.

8. $S_1(\mathcal{S}, \mathcal{S})$

In [29] (Theorem 2.5), M. Sakai proved:

Theorem 8.1. (*Sakai*) *For a space X, the following statements are equivalent:*

1. $C_p(X) \models S_1(\Gamma_x, \Gamma_x)$*;*
2. $X \models S_1(\Gamma_{cl}, \Gamma_{cl})$ *and is strongly zero-dimensional.*

Proposition 8.2. *For a strongly zero-dimensional space X, the following statements are equivalent:*

1. $X \models S_1(\Gamma_F, \Gamma)$*;*
2. $X \models S_1(\Gamma_{cl}, \Gamma_{cl})$*.*

Proposition 8.3. *For a space X, the following statements are equivalent:*

1. $C_p(X) \models S_1(\Gamma_x, \Gamma_x)$*;*
2. $X \models S_1(\Gamma_F, \Gamma)$*.*

Proof. $(1) \Rightarrow (2)$. By Theorem 8.1 and Proposition 8.2.

$(2) \Rightarrow (1)$. Let $X \models S_1(\Gamma_F, \Gamma)$ and $\{A_i\}_{i\in\omega}$ such that $A_i \in \Gamma_{\mathbf{0}}$ for each $i \in \omega$. Consider $\mathcal{U}_i = \{f^{-1}(-\frac{1}{i}, \frac{1}{i}) : f \in A_i\}$ for each $i \in \omega$. Without loss of generality we can assume that there is i' that a set $U \neq X$ for any $i > i'$ and $U \in \mathcal{U}_i$. Otherwise there is sequence $\{f_{i_k}\}_{k\in\omega}$ such that $\{f_{i_k}\}_{k\in\omega}$ uniform converge to $\mathbf{0}$ and $\{f_{i_k} : k \in \omega\} \in \Omega_{\mathbf{0}}$.

Note that $\mathcal{F}_i = \{f^{-1}[-\frac{1}{i+1}, \frac{1}{i+1}] : f \in A_i\}$ is γ-cover of zero-sets of X. It follows that $\mathcal{U}_i \in \Gamma_F$ for each $i \in \omega$. By $X \models S_1(\Gamma_F, \Gamma)$, there is a set $\{U_i : i \in \omega\}$ such that for each i, $U_i \in \mathcal{U}_i$, and $\{U_i : i \in \omega\}$ is an element of Γ.

We claim that $\{f_i : i \in \omega\} \in \Gamma_{\mathbf{0}}$. Let $W =< \mathbf{0}\ , K, \epsilon >$ be a base neighborhood of $\mathbf{0}$ where $\epsilon > 0$ and K is a finite subset of X, then there are $i_0 \in \omega$ such that $\frac{1}{i_0} < \epsilon$ and $U_i \supset K$ for any $i > i_0$. It follows that $f_i \in W$ for $i > i_0$, and $C_p(X) \models S_1(\Gamma_x, \Gamma_x)$.

By Theorem 6.1, we have that X is strongly zero-dimensional.

□

Theorem 8.4. *For a space X, the following statements are equivalent:*

1. $C_p(X) \models S_1(\mathcal{S}, \mathcal{S})$*;*
2. $X \models S_1(\Gamma_F, \Gamma)$*,* $X \models V$ *and is strongly zero-dimensional;*
3. $X \models S_1(\Gamma_{cl}, \Gamma_{cl})$*,* $X \models V$ *and is strongly zero-dimensional;*
4. $C_p(X) \models S_1(\Gamma_x, \Gamma_x)$ *and is sequentially separable;*
5. $C_p(X) \models S_1(\mathcal{S}, \Gamma_x)$ *and is sequentially separable.*

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{V}_i\} \subset \Gamma_F$ and $\mathcal{S} = \{h_m : m \in \omega\}$ be a countable sequentially dense subset of $C_p(X)$. For each $i \in \omega$ we consider a countable sequentially dense subset $\mathcal{S}_i$ of $C_p(X)$ and $\mathcal{U}_i = \{U_i^m : m \in \omega\} \subset \mathcal{V}_i$ where

$\mathcal{S}_i = \{f_i^m \in C(X) : f_i^m \upharpoonright F(U_i^m) = h_m$ and $f_i^m \upharpoonright (X \setminus U_i^m) = 1$ for $m \in \omega\}$. Note that $\mathcal{U}_i \in \Gamma_F$ for each $i \in \omega$.

Since $\mathcal{F}_i = \{F(U_i^m) : m \in \omega\}$ is infinity and it is a γ-cover of zero-sets of X. Since $\mathcal{S}$ is a countable sequentially dense subset of $C_p(X)$, we have that $\mathcal{S}_i$ is a countable sequentially dense subset of $C_p(X)$ for each $i \in \omega$. Let $h \in C(X)$, there is a set $\{h_{m_s} : s \in \omega\} \subset \mathcal{S}$ such that $\{h_{m_s}\}_{s\in\omega}$ converge to h. Let K be a finite subset of X, $\epsilon > 0$ and $W =< h, K, \epsilon >$ be a base neighborhood of h, then there is a number m_0 such that $K \subset F(U_i^m)$ for $m > m_0$ and $h_{m_s} \in W$ for $m_s > m_0$. Since $f_i^{m_s} \upharpoonright K = h_{m_s} \upharpoonright K$ for each $m_s > m_0$, $f_i^{m_s} \in W$ for each $m_s > m_0$. It follows that a sequence $\{f_i^{m_s}\}_{s\in\omega}$ converge to h.

Since $C(X) \models S_1(\mathcal{S}, \mathcal{S})$, there is a sequence $\{f_i^{m(i)}\}_{i\in\omega}$ such that for each i, $f_i^{m(i)} \in \mathcal{S}_i$, and $\{f_i^{m(i)} : i \in \omega\}$ is an element of $\mathcal{S}$.

Consider a set $\{U_i^{m(i)} : i \in \omega\}$.

(a). $U_i^{m(i)} \in \mathcal{U}_i$.

(b). $\{U_i^{m(i)} : i \in \omega\}$ is a γ-cover of X.

There is a sequence $\{f_{i_j}^{m(i)_j}\}$ converge to $\mathbf{0}$. Let K be a finite subset of X and $U =< \mathbf{0}\ , K, (-1, 1) >$ be a base neighborhood of $\mathbf{0}$, then there exists $j_0 \in \omega$ such that $f_{i_j}^{m(i)_j} \in U$ for each $j > j_0$. It follows that $K \subset U_{i_j}^{m(i)_j}$ for $j > j_0$. We thus get $X \models S_{fin}(\Gamma_F, \Gamma)$. But $S_{fin}(\Gamma_F, \Gamma) = S_1(\Gamma_F, \Gamma)$.

By Proposition 6.4, $X \models S_1(\Gamma_F, \Omega)$ implies $C_p(X) \models S_1(\Gamma_x, \Omega_x)$. By Theorem 6.1, X is strongly zero-dimensional.

(2) $\Rightarrow$ (1). Fix $\{S_i : i \in \omega\} \subset \mathcal{S}$ and $S = \{h_i : i \in \omega\} \in \mathcal{S}$. For each $i \in \omega$ we consider a set $\{f_k^i : k \in \omega\} \subset S_i$ such that $\{f_k^i\}_{k\in\omega}$ converge to h_i. For each $i, k \in \omega$, we put $U_{i,k} = \{x \in X : |f_k^i(x) - h_i(x)| < \frac{1}{i}\}$, $Z_{i,k} = \{x \in X : |f_k^i(x) - h_i(x)| \leq \frac{1}{i+1}\}$. Each $U_{i,k}$ (resp., $Z_{i,k}$) is a cozero-set (resp., zero-set)

in X with $Z_{i,k} \subset U_{i,k}$. Let $\mathcal{U}_i = \{U_{i,k} : k \in \omega\}$ and $\mathcal{Z}_i = \{Z_{i,k} : k \in \omega\}$. So without loss of generality, we may assume $U_{i,k} \neq X$ for each $i, k \in \omega$. We can easily check that the condition $f_k^i \to h_i$ $(k \to \infty)$ implies that $\mathcal{Z}_i$ is a γ-cover of X. Since $X \models S_1(\Gamma_F, \Gamma)$ there is a sequence $\{U_{i,k(i)}\}_{i\in\omega}$ such that for each i, $U_{i,k(i)} \in \mathcal{U}_i$, and $\{U_{i,k(i)} : i \in \omega\}$ is an element of Γ. We claim that $\{f_{k(i)}^i\}_{i\in\omega} \in \mathcal{S}$. For $f \in C(X)$ there is a set $\{h_{i_s} : s \in \omega\} \subset S$ such that $\{h_{i_s}\}_{s\in\omega}$ converge to f. Then a sequence $\{f_{k(i_s)}^{i_s}\}_{s\in\omega}$ converge to f too. Let K be a finite subset of X, $\epsilon > 0$, and $U =< f, K, \epsilon >$ be a base neighborhood of f, then there exists $m \in \omega$ such that $h_{i_s} \in< f, K, \frac{\epsilon}{2} >$ for each $i_s > m$. Since $\{U_{i,k(i)} : i \in \omega\}$ is an element of Γ, there exists $n > m$ such that $\frac{1}{n} < \frac{\epsilon}{2}$ and $K \subset U_{i,k(i)}$ for $i > n$. It follows that for each $i_s > n$ and $x \in K$ we have that $|f(x) - f_{k(i_s)}^{i_s}(x)| \leq |f(x) - h_{i_s}(x)| + |f_{k(i_s)}^{i_s}(x) - h_{i_s}(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence a sequence $\{f_{k(i_s)}^{i_s}\}_{s\in\omega}$ converge to f and $\{f_{k(i)}^i\}_{i\in\omega} \in \mathcal{S}$.

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). By Theorem 8.1, Proposition 8.2, Proposition 8.3 and Theorem 2.3.

(4) $\Rightarrow$ (5) is immediate.

(5) $\Rightarrow$ (2). Let $C_p(X) \models S_1(\mathcal{S}, \Gamma_x)$ and $C_p(X)$ be a sequentially separable.

Let $\{\mathcal{V}_i\} \subset \Gamma_F$ and $\mathcal{S} = \{h_m : m \in \omega\}$ be a countable sequentially dense subset of $C_p(X)$. For each $i \in \omega$ we consider a countable sequentially dense subset $\mathcal{S}_i$ of $C_p(X)$ and $\mathcal{U}_i = \{U_i^m\}_{m\in\omega} \subset \mathcal{V}_i$ where

$\mathcal{S}_i = \{f_i^m \in C(X) : f_i^m \upharpoonright F(U_i^m) = h_m$ and $f_i^m \upharpoonright (X \setminus U_i^m) = 1$ for $m \in \omega\}$.

Since $\mathcal{F}_i = \{F(U_i^m) : m \in \omega\}$ is infinity, it is a γ-cover of zero-subsets in X. Since $\mathcal{S}$ is a countable sequentially dense subset of $C_p(X)$, we have that $\mathcal{S}_i$ is a countable sequentially dense subset of $C_p(X)$ for each $i \in \omega$.

Let $h \in C(X)$, there is a set $\{h_{m_s} : s \in \omega\} \subset \mathcal{S}$ such that $\{h_{m_s}\}_{s\in\omega}$ converge to h. Let K be a finite subset of X, $\epsilon > 0$ and $W =< h, K, \epsilon >$ be a base neighborhood of h, then there is a number m_0 such that $K \subset F(U_i^m)$ for $m > m_0$ and $h_{m_s} \in W$ for $m_s > m_0$. Since $f_i^{m_s} \upharpoonright K = h_{m_s} \upharpoonright K$ for each $m_s > m_0$, $f_i^{m_s} \in W$ for each $m_s > m_0$. It follows that a sequence $\{f_i^{m_s}\}_{s\in\omega}$ converge to h.

By $C_p(X) \models S_1(\mathcal{S}, \Gamma_x)$, there is a sequence $\{f_i^{m(i)} : i \in \omega\}$ such that for each i, $f_i^{m(i)} \in \mathcal{S}_i$, and $\{f_i^{m(i)} : i \in \omega\}$ is an element of Γ_0.

Consider a set $\{U_i^{m(i)} : i \in \omega\}$.

(a). $U_i^{m(i)} \in \mathcal{U}_i$.

(b). $\{U_i^{m(i)} : i \in \omega\}$ is a γ-cover of X.

Let K be a finite subset of X and $U =< \mathbf{0}\ , K, \frac{1}{2} >$ be a base neighborhood of $\mathbf{0}$, then there is $j_0 \in \omega$ such that $f_{i_j}^{m(i)_j} \in U$ for each $j > j_0$. It follows that $K \subset U_{i_j}^{m(i)_j}$ for each $j > j_0$. We thus get $X \models S_1(\Gamma_F, \Gamma)$. By Theorem 2.3, $X \models V$. Since $C_p(X) \models S_1(\mathcal{S}, \Gamma_x)$ implies that $C_p(X) \models S_1(\mathcal{S}, \Omega_x)$, by Theorem 6.6, we have that X is strongly zero-dimensional.

□

Corollary 8.5. For a separable metrizable space X, the following statements are equivalent:

1. $C_p(X) \models S_1(\mathcal{S}, \mathcal{S})$;
2. $X \models S_1(\Gamma_F, \Gamma)$ and is strongly zero-dimensional;
3. $X \models S_1(\Gamma_{cl}, \Gamma_{cl})$ and is strongly zero-dimensional;
4. $C_p(X) \models S_1(\Gamma_x, \Gamma_x)$;
5. $C_p(X) \models S_1(\mathcal{S}, \Gamma_x)$.

The proof of fact that $S_{fin}(\Gamma_F, \Gamma) = S_1(\Gamma_F, \Gamma)$ (or $S_1(\Gamma_{cl}, \Gamma_{cl}) = S_{fin}(\Gamma_{cl}, \Gamma_{cl})$) is analogous to proof of Theorem 1.1 (in [16]) that $S_1(\Gamma, \Gamma) = S_{fin}(\Gamma, \Gamma)$.

Proposition 8.6. *For a space X, the following statements are equivalent:*

1. $C_p(X) \models S_{fin}(\Gamma_x, \Gamma_x)$*;*
2. $X \models S_{fin}(\Gamma_F, \Gamma)$*;*
3. $X \models S_{fin}(\Gamma_{cl}, \Gamma_{cl})$ *and is strongly zero-dimensional;*
4. $X \models S_1(\Gamma_F, \Gamma)$.

Proof. Note that, by Theorem 2 in [34], $C_p(X) \models S_1(\Gamma_x, \Gamma_x)$ iff $C_p(X) \models S_{fin}(\Gamma_x, \Gamma_x)$. By Theorem 8.1 and Proposition 8.2, we obtain the complete proof. □

Theorem 8.7. *For a space X, the following statements are equivalent:*

1. $C_p(X) \models S_{fin}(\mathcal{S}, \mathcal{S})$ *and is sequentially separable;*
2. $X \models S_1(\Gamma_F, \Gamma)$, $X \models V$ *and is strongly zero-dimensional;*
3. $X \models S_{fin}(\Gamma_F, \Gamma)$, $X \models V$ *and is strongly zero-dimensional;*
4. $X \models S_{fin}(\Gamma_{cl}, \Gamma_{cl})$, $X \models V$ *and is strongly zero-dimensional;*
5. $C_p(X) \models S_{fin}(\Gamma_x, \Gamma_x)$ *and is sequentially separable;*
6. $C_p(X) \models S_{fin}(\mathcal{S}, \Gamma_x)$ *and is sequentially separable.*

Proof. By Proposition 8.6, Theorem 8.4 and Theorem 2.3.

□

Corollary 8.8. For a separable metrizable space X, the following statements are equivalent:

1. $C_p(X) \models S_{fin}(\mathcal{S}, \mathcal{S})$;
2. $X \models S_1(\Gamma_F, \Gamma)$, and is strongly zero-dimensional;
3. $X \models S_{fin}(\Gamma_F, \Gamma)$, and is strongly zero-dimensional;
4. $X \models S_{fin}(\Gamma_{cl}, \Gamma_{cl})$, and is strongly zero-dimensional;
5. $C_p(X) \models S_{fin}(\Gamma_x, \Gamma_x)$;
6. $C_p(X) \models S_{fin}(\mathcal{S}, \Gamma_x)$.

9. $U_{fin}(\mathcal{S}, \mathcal{D})$

Recall that $U_{fin}(\mathcal{S}, \mathcal{D})$ is the selection hypothesis: whenever $\mathcal{U}_1, \mathcal{U}_2, ... \in \mathcal{S}$, there are finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \omega$, such that $\{\bigcup \mathcal{F}_n : n \in \omega\} \in \mathcal{D}$. For a function space $C(X)$, we can represent the condition $\{\bigcup \mathcal{F}_n : n \in \omega\} \in \mathcal{D}$ as $\forall\ f \in C(X)$ $\forall$ a base neighborhood $O(f) =< f, K, \epsilon >$ of f where $\epsilon > 0$ and $K = \{x_1, ..., x_k\}$ is a finite subset of X, there is $n' \in \omega$ such that for each $j \in \{1, ..., k\}$ there is $g \in \mathcal{F}_{n'}$ such that $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$.

Similarly, $U_{fin}(\Gamma_0, \Omega_0)$: whenever $\mathcal{S}_1$, $\mathcal{S}_2, ... \in \Gamma_0$, there are finite sets $\mathcal{F}_n \subseteq \mathcal{S}_n$, $n \in \omega$, such that $\{\bigcup \mathcal{F}_n : n \in \omega\} \in \Omega_0$, i.e. for a base neighborhood $O(f) =< f, K, \epsilon >$ of $f = \mathbf{0}$ where $\epsilon > 0$ and $K = \{x_1, ..., x_k\}$ is a finite subset of X, there is $n' \in \omega$ such that for each $j \in \{1, ..., k\}$ there is $g \in \mathcal{F}_{n'}$ such that $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$.

Theorem 9.1. *For a space X, the following statements are equivalent:*

1. $C_p(X) \models U_{fin}(\Gamma_x, \Omega_x)$*;*
2. $X \models U_{fin}(\Gamma_F, \Omega)$.

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{V}_i\} \subset \Gamma_F$. For each $i \in \omega$ and $\mathcal{U}_i = \{U_i^m\} \subset \mathcal{V}_i$ we consider $\mathcal{K}_i = \{f_i^m \in C(X) : f_i^m \upharpoonright F(U_i^m) = 0$ and $f_i^m \upharpoonright (X \setminus U_i^m) = 1$ for $m \in \omega\}$. Note that $\mathcal{U}_i \in \Gamma_F$ for each $i \in \omega$.

Since $\mathcal{F}_i = \{F(U_i^m) : m \in \omega\}$ is a γ-cover of zero-sets of X, we have that $\mathcal{K}_i$ converge to $\mathbf{0}$ for each $i \in \omega$. By $C_p(X) \models U_{fin}(\Gamma_x, \Omega_x)$, there are finite sets $F_i = \{f_i^{m_1}, ..., f_i^{m_{s(i)}}\} \subseteq \mathcal{K}_i$ such that $\{\bigcup F_i : i \in \omega\} \in \Omega_0$. Note that $\{\bigcup\{U_i^{m_1}, ..., U_i^{m_{s(i)}}\} : i \in \omega\} \in \Omega$.

(2) $\Rightarrow$ (1). Let $X \models U_{fin}(\Gamma_F, \Omega)$ and $A_i \in \Gamma_{\mathbf{0}}$ for each $i \in \omega$. Consider $\mathcal{U}_i = \{U_{i,f} = f^{-1}(-\frac{1}{i}, \frac{1}{i}) : f \in A_i\}$ for each $i \in \omega$. Without loss of generality we can assume that a set $U_{i,f} \neq X$ for any $i \in \omega$ and $f \in A_i$. Otherwise

there is sequence $\{f_{i_k}\}_{k\in\omega}$ such that $\{f_{i_k}\}_{k\in\omega}$ uniform converge to $\mathbf{0}$ and $\{f_{i_k} : k \in \omega\} \in \Omega_{\mathbf{0}}$.

Note that $\mathcal{F}_i = \{F_{i,m}\}_{m\in\omega} = \{f_{i,m}^{-1}[-\frac{1}{i+1}, \frac{1}{i+1}] : m \in \omega\}$ is γ-cover of zero-sets of X and $F_{i,m} \subset U_{i,m}$ for each $i, m \in \omega$. It follows that $\mathcal{U}_i \in \Gamma_F$ for each $i \in \omega$.

By $X \models U_{fin}(\Gamma_F, \Omega)$, there is a sequence $\{U_{i,m(1)}, U_{i,m(2)}, ..., U_{i,m(i)} : i \in \omega\}$ such that for each i and $k \in \{m(1), ..., m(i)\}$, $U_{i,m(k)} \in \mathcal{U}_i$, and

$\{\bigcup\{U_{i,m(1)}, ..., U_{i,m(i)}\} : i \in \omega\} \in \Omega$.

We claim that $\{\bigcup\{f_{i,m(1)}, ..., f_{i,m(i)}\} : i \in \omega\} \in \Omega_0$.

Let $W =< \mathbf{0}\ , K, \epsilon >$ be a base neighborhood of $\mathbf{0}$ where $\epsilon > 0$ and $K = \{x_1, ..., x_s\}$ is a finite subset of X, then there are $i_0, i_1 \in \omega$ such that $\frac{1}{i_0} < \epsilon$, $i_1 > i_0$ and $\bigcup_{k=m(1)}^{m(i_1)} U_{i_1,k} \supset K$. It follows that for each $j \in \{1, ..., s\}$ there is $g \in \{f_{i_1,m(1)}, ..., f_{i_1,m(i_1)}\}$ such that $g(x_j) \in (-\epsilon, \epsilon)$.

□

Theorem 9.2. *For a space X, the following statements are equivalent:*

1. $C_p(X) \models U_{fin}(\mathcal{S}, \mathcal{D})$ *and is sequentially separable;*
2. $X \models U_{fin}(\Gamma_F, \Omega)$, $X \models V$*;*
3. $C_p(X) \models U_{fin}(\Gamma_x, \Omega_x)$ *and is sequentially separable;*
4. $C_p(X) \models U_{fin}(\mathcal{S}, \Omega_x)$ *and is sequentially separable.*

Proof. (1) ⇒ (2). Suppose that $C_p(X) \models U_{fin}(\mathcal{S}, \mathcal{D})$ and is sequentially separable. Let $\{\mathcal{V}_i\} \subset \Gamma_F$ and $\mathcal{S} = \{h_j : j \in \omega\}$ be a countable sequentially dense subset of $C_p(X)$.

For each $i \in \omega$ and $\mathcal{U}_i = \{U_i^j : j \in \omega\} \subset \mathcal{V}_i$ we consider $\mathcal{S}_i = \{f_i^j \in C(X) : f_i^j \upharpoonright F(U_i^j) = h_j$ and $f_i^j \upharpoonright (X \setminus U_i^j) = 1$ for $j \in \omega\}$.

Since $\mathcal{F}_i = \{F(U_i^m) : m \in \omega\}$ is a γ-cover of X, we have that $\mathcal{S}_i$ is a countable sequentially dense subset of $C_p(X)$ for each $i \in \omega$.

By $C_p(X) \models U_{fin}(\mathcal{S}, \mathcal{D})$, there are finite sets $F_i = \{f_i^{m_1}, ..., f_i^{m_{s(i)}}\} \subseteq \mathcal{S}_i$ such that $\{\bigcup F_i : i \in \omega\} \in \mathcal{D}$. Note that $\{\bigcup\{U_i^{m_1}, ..., U_i^{m_{s(i)}}\} : i \in \omega\} \in \Omega$. By Theorem 2.3, $X \models V$.

(2) ⇒ (3). By Theorem 2.3 and Theorem 9.1.

(3) ⇒ (4) is immediate.

(4) ⇒ (1). Suppose that $C_p(X)$ is sequentially separable and $C_p(X) \models U_{fin}(\mathcal{S}, \Omega_x)$.

Let $D = \{d_n : n \in \omega\}$ be a dense subspace of $C_p(X)$. Given a sequence of sequentially dense subspace of $C_p(X)$, enumerate it as $\{S_{n,m} : n, m \in \omega\}$. For each $n \in \omega$, pick

$\mathcal{F}_{n,m} = \{d_{n,m,1}, ..., d_{n,m,k(n,m)}\} \subset S_{n,m}$ so that $d_n \in \overline{\{\bigcup \mathcal{F}_{n,m} : m \in \omega\}}$, i.e. for a base neighborhood $O(d_n) =< d_n, K, \epsilon >$ of d_n where $\epsilon > 0$ and $K = \{x_1, ..., x_k\}$ is a finite subset of X, there is $m' \in \omega$ such that for each $j \in \{1, ..., k\}$ there is $g \in \mathcal{F}_{n,m'}$ such that $g(x_j) \in (d_n(x_j) - \epsilon, d_n(x_j) + \epsilon)$.

Then $\{\bigcup \mathcal{F}_{n,m} : m, n \in \omega\} \in \mathcal{D}$.

□

Theorem 9.3. *For a separable metrizable space X, the following statements are equivalent:*

1. $C_p(X) \models U_{fin}(\mathcal{S}, \mathcal{D})$*;*
2. $X \models U_{fin}(\Gamma, \Omega)$*;*
3. $C_p(X) \models U_{fin}(\Gamma_x, \Omega_x)$*;*
4. $C_p(X) \models U_{fin}(\mathcal{S}, \Omega_x)$*.*

10. $U_{fin}(\mathcal{S}, \mathcal{S})$

Recall that $U_{fin}(\mathcal{S}, \mathcal{S})$ is the selection hypothesis: whenever $\mathcal{U}_1, \mathcal{U}_2, ... \in \mathcal{S}$, there are finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \omega$, such that $\{\bigcup \mathcal{F}_n : n \in \omega\} \in \mathcal{S}$. For a function space $C(X)$, we can represent the condition $\{\bigcup \mathcal{F}_n : n \in \omega\} \in \mathcal{S}$ as $\forall\ f \in C(X)\ \forall$ a base neighborhood of f $O(f) =< f, K, \epsilon >$ of f where $\epsilon > 0$ and $K = \{x_1, ..., x_k\}$ is a finite subset of X, there is $n' \in \omega$ such that for each $n > n'$ and $j \in \{1, ..., k\}$ there is $g \in \mathcal{F}_n$ such that $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$.

Similarly, $U_{fin}(\Gamma_0, \Gamma_0)$: whenever $\mathcal{S}_1$, $\mathcal{S}_2, ... \in \Gamma_0$, there are finite sets $\mathcal{F}_n \subseteq \mathcal{S}_n$, $n \in \omega$, such that $\{\bigcup \mathcal{F}_n : n \in \omega\} \in \Gamma_0$, i.e. for a base neighborhood $O(f) =< f, K, \epsilon >$ of $f = \mathbf{0}$ where $\epsilon > 0$ and $K = \{x_1, ..., x_k\}$ is a finite subset of X, there is $n' \in \omega$ such that for each $n > n'$ and $j \in \{1, ..., k\}$ there is $g \in \mathcal{F}_n$ such that $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$.

Theorem 10.1. *For a space X, the following statements are equivalent:*

1. $C_p(X) \models U_{fin}(\Gamma_x, \Gamma_x)$*;*
2. $X \models U_{fin}(\Gamma_F, \Gamma)$*.*

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{V}_i\} \subset \Gamma_F$. For each $i \in \omega$ we consider a subset $\mathcal{S}_i$ of $C_p(X)$ and $\mathcal{U}_i = \{U_i^m\}_{m \in \omega} \subset \mathcal{V}_i$ where

$\mathcal{S}_i = \{f_i^m \in C(X) : f_i^m \upharpoonright F(U_i^m) = 0$ and $f_i^m \upharpoonright (X \setminus U_i^m) = 1$ for $m \in \omega\}$.

Since $\mathcal{F}_i = \{F(U_i^m) : m \in \omega\}$ is a γ-cover of X, we have that $\mathcal{S}_i$ converge to $\mathbf{0}$, i.e. $\mathcal{S}_i \in \Gamma_0$ for each $i \in \omega$.

Since $C(X) \models U_{fin}(\Gamma_x, \Gamma_x)$, there is a sequence $\{\mathcal{F}_i\}_{i\in\omega} = \{f_i^{m_1}, ..., f_i^{m_{k(i)}} : i \in \omega\}$ such that for each i, $\mathcal{F}_i \subseteq \mathcal{S}_i$, and $\{\bigcup \mathcal{F}_i\}_{i\in\omega} \in \Gamma_0$.

Consider a sequence $\{W_i\}_{i\in\omega} = \{U_i^{m_1}, ..., U_i^{m_{k(i)}} : i \in \omega\}$.

(a). $W_i \subset \mathcal{U}_i$.

(b). $\{\bigcup W_i : i \in \omega\}$ is a γ-cover of X.

Let $K = \{x_1, ..., x_s\}$ be a finite subset of X and $U =< \mathbf{0}\ , K, \frac{1}{2} >$ be a base neighborhood of $\mathbf{0}$, then there exists $i_0 \in \omega$ such that for each $i > i_0$ and

$j \in \{1, ..., s\}$ there is $g \in \mathcal{F}_i$ such that $g(x_j) \in (-\frac{1}{2}, \frac{1}{2})$.

It follows that $K \subset \bigcup\limits_{j=1}^{k(i)} U_i^{m_j}$ for $i > i_0$. We thus get $X \models U_{fin}(\Gamma_F, \Gamma)$.

(2) $\Rightarrow$ (1). Fix $\{S_i : i \in \omega\} \subset \Gamma_0$ where $S_i = \{f_k^i : k \in \omega\}$ for each $i \in \omega$.

For each $i, k \in \omega$, we put $U_{i,k} = \{x \in X : |f_k^i(x)| < \frac{1}{i}\}$, $Z_{i,k} = \{x \in X : |f_k^i(x)| \leq \frac{1}{i+1}\}$.

Each $U_{i,k}$ (resp., $Z_{i,k}$) is a cozero-set (resp., zero-set) in X with $Z_{i,k} \subset U_{i,k}$. Let $\mathcal{U}_i = \{U_{i,k} : k \in \omega\}$ and $\mathcal{Z}_i = \{Z_{i,k} : k \in \omega\}$. So without loss of generality, we may assume $U_{i,k} \neq X$ for each $i, k \in \omega$. We can easily check that the condition $f_k^i \to \mathbf{0}$ $(k \to \infty)$ implies that $\mathcal{Z}_i$ is a γ-cover of X.

Since $X \models U_{fin}(\Gamma_F, \Gamma)$ there is a sequence $\{\mathcal{F}_i\}_{i\in\omega} = \{U_{i,k_1}, ..., U_{i,k_i} : i \in \omega\}$ such that for each i, $\mathcal{F}_i \subset \mathcal{U}_i$, and $\{\bigcup \mathcal{F}_i : i \in \omega\}$ is an element of Γ.

Let $K = \{x_1, ..., x_s\}$ be a finite subset of X, $\epsilon > 0$, and $U =< \mathbf{0}, K, \epsilon >$ be a base neighborhood of $\mathbf{0}$, then there exists $i' \in \omega$ such that for each $i > i'$ $K \subset \bigcup \mathcal{F}_i$. It follow that for each $i > i'$ and $j \in \{1, ..., s\}$ there is $g \in \mathcal{S}_i$ such that $g(x_j) \in (-\epsilon, \epsilon)$. So $C_p(X) \models U_{fin}(\Gamma_x, \Gamma_x)$.

□

Theorem 10.2. *For a space X, the following statements are equivalent:*

1. $C_p(X) \models U_{fin}(\mathcal{S}, \mathcal{S})$ *and is sequentially separable;*
2. $X \models U_{fin}(\Gamma_F, \Gamma)$, $X \models V$*;*
3. $C_p(X) \models U_{fin}(\Gamma_x, \Gamma_x)$ *and is sequentially separable;*
4. $C_p(X) \models U_{fin}(\mathcal{S}, \Gamma_x)$ *and is sequentially separable.*

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{V}_i\} \subset \Gamma_F$ and $\mathcal{S} = \{h_m : m \in \omega\}$ be a countable sequentially dense subset of $C_p(X)$. For each $i \in \omega$ we consider a countable sequentially dense subset $\mathcal{S}_i$ of $C_p(X)$ and $\mathcal{U}_i = \{U_i^m\}_{m\in\omega} \subset \mathcal{V}_i$ where

$\mathcal{S}_i = \{f_i^m \in C(X) : f_i^m \upharpoonright F(U_i^m) = h_m$ and $f_i^m \upharpoonright (X \setminus U_i^m) = 1$ for $m \in \omega\}$.

Since $\mathcal{F}_i = \{F(U_i^m) : m \in \omega\}$ is a γ-cover of zero-sets of X and $\mathcal{S}$ is a countable sequentially dense subset of $C_p(X)$, we have that $\mathcal{S}_i$ is a countable sequentially dense subset of $C_p(X)$ for each $i \in \omega$. Let $h \in C(X)$, there is a sequence $\{h_{m_s} : s \in \omega\} \subset \mathcal{S}$ such that $\{h_{m_s}\}_{s\in\omega}$ converge to h. Let K be a finite subset of X, $\epsilon > 0$ and $W =< h, K, \epsilon >$ be a base neighborhood of h, then there is a number m_0 such that $K \subset F(U_i^m)$ for $m > m_0$ and $h_{m_s} \in W$ for $m_s > m_0$. Since $f_i^{m_s} \upharpoonright K = h_{m_s} \upharpoonright K$ for each $m_s > m_0$, $f_i^{m_s} \in W$ for each $m_s > m_0$. It follows that a sequence $\{f_i^{m_s}\}_{s\in\omega}$ converge to h.

Since $C(X) \models U_{fin}(\mathcal{S}, \mathcal{S})$, there is a sequence $\{F_i\} = \{f_i^{m_1}, ..., f_i^{m_s} : i \in \omega\}$ such that for each i, $F_i \subset \mathcal{S}_i$, and $\{\bigcup F_i : i \in \omega\}$ is an element of $\mathcal{S}$, i.e. for any $f \in C(X)$ and a base neighborhood $O(f) =< f, K, \epsilon >$ of f where $\epsilon > 0$ and $K = \{x_1, ..., x_k\}$ is a finite subset of X, there is $i' \in \omega$ such that for each $i > i'$ and $j \in \{1, ..., k\}$ there is $g \in \mathcal{F}_i$ such that $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$.

Consider a sequence $\{Q_i\}_{i\in\omega} = \{U_i^{m_1}, ..., U_i^{m_s} : i \in \omega\}$.

(a). $Q_i \subset \mathcal{U}_i$.

(b). $\{\bigcup Q_i : i \in \omega\}$ is a γ-cover of X.

We thus get $X \models U_{fin}(\Gamma_F, \Gamma)$. By Theorem 2.3, $X \models V$.

□

Theorem 10.3. *For a separable metrizable space X, the following statements are equivalent:*

1. $C_p(X) \models U_{fin}(\mathcal{S}, \mathcal{S})$;
2. $X \models U_{fin}(\Gamma, \Gamma)$ *[Hurewicz property];*
3. $C_p(X) \models U_{fin}(\Gamma_x, \Gamma_x)$;
4. $C_p(X) \models U_{fin}(\mathcal{S}, \Gamma_x)$.

11. $S_1(\mathcal{A}, \mathcal{A})$

Definition 11.1. A set $A \subseteq C_p(X)$ will be called *n-dense* in $C_p(X)$, if for each n-finite set $\{x_1, ..., x_n\} \subset X$ such that $x_i \neq x_j$ for $i \neq j$ and an open sets $W_1, ..., W_n$ in $\mathbb{R}$ there is $f \in A$ such that $f(x_i) \in W_i$ for $i \in \overline{1, n}$.

Obviously, that if A is a n-dense set of $C_p(X)$ for each $n \in \omega$ then A is a dense set of $C_p(X)$.

For a space $C_p(X)$ we denote:

$\mathcal{A}_n$ — the family of a n-dense subsets of $C_p(X)$.

If $n = 1$, then we denote $\mathcal{A}$ instead of $\mathcal{A}_1$.

Definition 11.2. Let $f \in C(X)$. A set $B \subseteq C_p(X)$ will be called *n-dense* at point f, if for each n-finite set $\{x_1, ..., x_n\} \subset X$ and $\epsilon > 0$ there is $h \in B$ such that $h(x_i) \in (f(x_i) - \epsilon, f(x_i) + \epsilon)$ for $i \in \overline{1, n}$.

Obviously, that if B is a n-dense at point f for each $n \in \omega$ then $f \in \overline{B}$.

For a space $C_p(X)$ we denote:

$\mathcal{B}_{n,f}$ — the family of a n-dense at point f subsets of $C_p(X)$.

If $n = 1$, then we denote $\mathcal{B}_f$ instead of $\mathcal{B}_{1,f}$.

Let $\mathcal{U}$ be an open cover of X and $n \in \mathbb{N}$.

• $\mathcal{U}$ is an n-cover of X if for each $F \subset X$ with $|F| \leq n$, there is $U \in \mathcal{U}$ such that $F \subset U$ [38].

• $\mathcal{O}_n$ — the family of open n-covers of X.

• $S_1(\mathcal{O}, \mathcal{O}) = S_1(\Omega, \mathcal{O})$ [32].

• $S_1(\Omega, \mathcal{O}) = S_1(\{\mathcal{O}_n\}_{n \in \mathbb{N}}, \mathcal{O})$ [38].

Theorem 11.3. *For a space X, the following statements are equivalent:*

1. $C_p(X) \models S_1(\mathcal{A}, \mathcal{A})$*;*
2. $X \models S_1(\mathcal{O}, \mathcal{O})$ *[Rothberger property];*
3. $C_p(X) \models S_1(\mathcal{B}_f, \mathcal{B}_f)$*;*
4. $C_p(X) \models S_1(\mathcal{A}, \mathcal{B}_f)$*;*
5. $C_p(X) \models S_1(\mathcal{D}, \mathcal{A})$*;*
6. $C_p(X) \models S_1(\{\mathcal{A}_n\}_{n \in \mathbb{N}}, \mathcal{A})$*;*
7. $C_p(X) \models S_1(\{\mathcal{B}_{n,f}\}_{n \in \mathbb{N}}, \mathcal{B}_f)$*;*
8. $C_p(X) \models S_1(\{\mathcal{A}_n\}_{n \in \mathbb{N}}, \mathcal{B}_f)$*.*

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{O}_n\}_{n \in \omega}$ be a sequence of open covers of X. We set $A_n = \{f \in C(X) : f \upharpoonright (X \setminus U) = 1$ and $f \upharpoonright K = q$ for some $U \in \mathcal{O}_n$, a finite set $K \subset U$ and $q \in \mathbb{Q}\}$. It is not difficult to see that each A_n is 1-dense subset of $C_p(X)$ since each $\mathcal{O}_n$ is a cover of X and X is Tychonoff.

By the assumption there exist $f_n \in A_n$ such that $\{f_n : n \in \omega\} \in \mathcal{A}$.

For each f_n we take $U_n \in \mathcal{O}_n$ such that $f_n \upharpoonright (X \setminus U_n) = 1$.

Set $\mathcal{U} = \{U_n : n \in \omega\}$. For $x \in X$ we consider the basic open neighborhood of $\mathbf{0}$ $[x, W]$, where $W = (-\frac{1}{2}, \frac{1}{2})$.

Note that there is $m \in \omega$ such that $[x, W]$ contains $f_m \in \{f_n : n \in \omega\}$. This means $x \in U_m$. Consequently $\mathcal{U}$ is cover of X.

(2) $\Rightarrow$ (3). Let $B_n \in \mathcal{B}_f$ for each $n \in \omega$. We renumber $\{B_n\}_{n \in \omega}$ as $\{B_{i,j}\}_{i,j \in \omega}$. Since $C(X)$ is homogeneous, we may think that $f = \mathbf{0}$. We set

$\mathcal{U}_{i,j} = \{g^{-1}(-1/i, 1/i) : g \in B_{i,j}\}$ for each $i, j \in \omega$. Since $B_{i,j} \in \mathcal{B}_0$, $\mathcal{U}_{i,j}$ is an open cover of X for each $i, j \in \omega$. In case the set $M = \{i \in \omega : X \in \mathcal{U}_{i,j}\}$ is infinite, choose $g_m \in B_{m,j}$ $m \in M$ so that $g^{-1}(-1/m, 1/m) = X$, then $\{g_m : m \in \omega\} \in \mathcal{B}_f$.

So we may assume that there exists $i' \in \omega$ such that for each $i \geq i'$ and $g \in B_{i,j}$ $g^{-1}(-1/i, 1/i)$ is not X.

For the sequence $\mathcal{V}_i = \{\mathcal{U}_{i,j} : j \in \omega\}$ of open covers there exist $f_{i,j} \in B_{i,j}$ such that $\mathcal{U}_i = \{f_{i,j}^{-1}(-1/i, 1/i) : j \in \omega\}$ is a cover of X. Let $[x, W]$ be any basic open neighborhood of $\mathbf{0}$, where $W = (-\epsilon, \epsilon)$, $\epsilon > 0$. There exists $m \geq i'$ and $j \in \omega$ such that $1/m < \epsilon$ and $x \in f_{m,j}^{-1}(-1/m, 1/m)$. This means $\{f_{i,j} : i, j \in \omega\} \in \mathcal{B}_f$.

(3) $\Rightarrow$ (4) is immediate.

(4) $\Rightarrow$ (1). Let $A_n \in \mathcal{A}$ for each $n \in \omega$. We renumber $\{A_n\}_{n\in\omega}$ as $\{A_{i,j}\}_{i,j\in\omega}$. Renumber the rational numbers $\mathbb{Q}$ as $\{q_i : i \in \omega\}$. Fix $i \in \omega$. By the assumption there exist $f_{i,j} \in A_{i,j}$ such that $\{f_{i,j} : j \in \omega\} \in \mathcal{A}_{q_i}$ where q_i is the constant function to q_i. Then $\{f_{i,j} : i, j \in \omega\} \in \mathcal{A}$.

(1) $\Rightarrow$ (5). Since a dense set of $C_p(X)$ is a 1-dense set of $C_p(X)$, we have $C_p(X) \models S_1(\mathcal{D}, \mathcal{A})$.

(5) $\Rightarrow$ (6). Let $D_n \in \mathcal{A}_n$ for each $n \in \omega$. We renumber $\{D_n\}_{n\in\omega}$ as $\{D_{i,j}\}_{i,j\in\omega}$. Then $P_j = \{D_{i,j} : i \in \omega\}$ is a dense subset of $C_p(X)$ for each $j \in \omega$. By (5), there is $p_j \in P_j$ for each $j \in \omega$ such that $\{p_j : j \in \omega\} \in \mathcal{A}$. Hence, we have $C_p(X) \models S_1(\{\mathcal{A}_n\}_{n\in\mathbb{N}}, \mathcal{A})$.

(6) $\Rightarrow$ (8) is immediate.

(8) $\Rightarrow$ (2). Claim that $X \models S_1(\{\mathcal{O}_n\}_{n\in\omega}, \mathcal{O})$. Fix $\{\mathcal{O}_n\}_{n\in\omega}$. For every $n \in \omega$ a set $\mathcal{S}_n = \{f \in C(X) : f \upharpoonright (X \setminus U) = 1$ and $f(x_i) \in \mathbb{Q}$ for each $i = \overline{1, n}$ for $U \in \mathcal{O}_n$ and a finite set $K = \{x_1, ..., x_n\} \subset U\}$. Note that $\mathcal{S}_n \in \mathcal{A}_n$ for each $n \in \omega$. By (8), there is $f_n \in \mathcal{S}_n$ for each $n \in \omega$ such that $\{f_n : n \in \omega\} \in \mathcal{B}_0$. Then $\{U_n : n \in \omega\} \in \mathcal{O}$.

(3) $\Rightarrow$ (7) is immediate.

(7) $\Rightarrow$ (2). The proof is analogous to proof of implication (8) $\Rightarrow$ (2).

$\square$

12. $S_{fin}(\mathcal{A}, \mathcal{A})$

Theorem 12.1. *For a space X, the following statements are equivalent:*

1. $C_p(X) \models S_{fin}(\mathcal{A}, \mathcal{A})$;
2. $X \models S_{fin}(\mathcal{O}, \mathcal{O})$ *[Menger property];*

3. $C_p(X) \models S_{fin}(\mathcal{B}_f, \mathcal{B}_f)$;
4. $C_p(X) \models S_{fin}(\mathcal{A}, \mathcal{B}_f)$;
5. $C_p(X) \models S_{fin}(\mathcal{D}, \mathcal{A})$;
6. $C_p(X) \models S_{fin}(\{\mathcal{A}_n\}_{n\in\mathbb{N}}, \mathcal{A})$;
7. $C_p(X) \models S_{fin}(\{\mathcal{B}_{n,f}\}_{n\in\mathbb{N}}, \mathcal{B}_f)$;
8. $C_p(X) \models S_{fin}(\{\mathcal{A}_n\}_{n\in\mathbb{N}}, \mathcal{B}_f)$.

Proof. The proof is analogous to proof of Theorem 11.3.

□

13. $S_1(\mathcal{S}, \mathcal{A})$

Proposition 13.1. *For a space X, the following statements are equivalent:*

1. $C_p(X) \models S_1(\Gamma_x, \mathcal{B}_f)$;
2. $X \models S_1(\Gamma_F, \mathcal{O})$.

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{U}_i\} \subset \Gamma_F$. For each $i \in \omega$ we consider the set $\mathcal{S}_i = \{f \in C(X) : f \upharpoonright F(U) = 0$ and $f \upharpoonright (X \setminus U) = 1$ for $U \in \mathcal{U}_i\}$.

Since $\mathcal{F}_i = \{F(U) : U \in \mathcal{U}_i\}$ is a γ-cover of X, we have that $\mathcal{S}_i$ converge to $\mathbf{0}$, i.e. $\mathcal{S}_i \in \Gamma_0$ for each $i \in \omega$.

Since $C_p(X) \models S_1(\Gamma_x, \mathcal{B}_f)$, there is a sequence $\{f_i\}_{i\in\omega}$ such that for each i, $f_i \in \mathcal{S}_i$, and $\{f_i : i \in \omega\} \in \mathcal{B}_{\mathbf{0}}$.

Consider $\mathcal{V} = \{U_i : U_i \in \mathcal{U}_i$ such that $f_i \upharpoonright F(U_i) = 0$ and $f_i \upharpoonright (X \setminus U_i) = 1\}$. Let $x \in X$ and $W = [x, (-1, 1)]$ be a neighborhood of $\mathbf{0}$, then there exists $i_0 \in \omega$ such that $f_{i_0} \in W$.

It follows that $x \in U_{i_0}$ and $\mathcal{V} \in \mathcal{O}$. We thus get $X \models S_1(\Gamma_F, \mathcal{O})$.

(2) $\Rightarrow$ (1). Fix $\{S_n : n \in \omega\} \subset \Gamma_0$. We renumber $\{S_n : n \in \omega\}$ as $\{S_{i,j} : i, j \in \omega\}$.

For each $i, j \in \omega$ and $f \in S_{i,j}$, we put $U_{i,j,f} = \{x \in X : |f(x)| < \frac{1}{i+j}\}$, $Z_{i,j,f} = \{x \in X : |f(x)| \leq \frac{1}{i+j+1}\}$.

Each $U_{i,j,f}$ (resp., $Z_{i,j,f}$) is a cozero-set (resp., zero-set) in X with $Z_{i,j,f} \subset U_{i,j,f}$. Let $\mathcal{U}_{i,j} = \{U_{i,j,f} : f \in S_{i,j}\}$ and $\mathcal{Z}_{i,j} = \{Z_{i,j,f} : f \in S_{i,j}\}$. So without loss of generality, we may assume $U_{i,j,f} \neq X$ for each $i, j \in \omega$ and $f \in S_{i,j}$.

We can easily check that the condition $S_{i,j} \in \Gamma_0$ implies that $\mathcal{Z}_{i,j}$ is a γ-cover of X.

Since $X \models S_1(\Gamma_F, \mathcal{O})$ for each $j \in \omega$ there is a sequence $\{U_{i,j,f_{i,j}} : i \in \omega\}$ such that for each i, $U_{i,j,f_{i,j}} \in \mathcal{U}_{i,j}$, and $\{U_{i,j,f_{i,j}} : i \in \omega\} \in \mathcal{O}$. Claim that

$\{f_{i,j} : i, j \in \omega\} \in \mathcal{B}_0$. Let $x \in X$, $\epsilon > 0$, and $W = [x, (-\epsilon, \epsilon)]$ be a base neighborhood of $\mathbf{0}$, then there exists $j' \in \omega$ such that $\frac{1}{1+j'} < \epsilon$. It follow that there exists i' such that $f_{i',j'}(x) \in (-\epsilon, \epsilon)$. So $C_p(X) \models S_1(\Gamma_x, \mathcal{B}_f)$.

$\square$

Theorem 13.2. *For a space X, the following statements are equivalent:*

1. $C_p(X) \models S_1(\mathcal{S}, \mathcal{A})$ *and is sequentially separable;*
2. $X \models S_1(\Gamma_F, \mathcal{O})$*,* $X \models V$*;*
3. $C_p(X) \models S_1(\Gamma_x, \mathcal{B}_f)$ *and is sequentially separable;*
4. $C_p(X) \models S_1(\mathcal{S}, \mathcal{B}_f)$ *and is sequentially separable.*

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{V}_i : i \in \omega\} \subset \Gamma_F$ and $\mathcal{S} = \{h_m : m \in \omega\}$ be a countable sequentially dense subset of $C_p(X)$. For each $i \in \omega$ we consider a countable sequentially dense subset $\mathcal{S}_i$ of $C_p(X)$ and $\mathcal{U}_i = \{U_i^m : m \in \omega\} \subset \mathcal{V}_i$ where

$\mathcal{S}_i = \{f_i^m \in C(X) : f_i^m \upharpoonright F(U_i^m) = h_m$ and $f_i^m \upharpoonright (X \setminus U_i^m) = 1$ for $m \in \omega\}$.

Since $\mathcal{F}_i = \{F(U_i^m) : m \in \omega\}$ is a γ-cover of zero subsets of X and $\mathcal{S}$ is a countable sequentially dense subset of $C_p(X)$, we have that $\mathcal{S}_i$ is a countable sequentially dense subset of $C_p(X)$ for each $i \in \omega$. Let $h \in C(X)$, there is a sequence $\{h_{m_s} : s \in \omega\} \subset \mathcal{S}$ such that $\{h_{m_s}\}_{s\in\omega}$ converge to h. Let K be a finite subset of X, $\epsilon > 0$ and $W =< h, K, \epsilon >$ be a base neighborhood of h, then there is a number m_0 such that $K \subset F(U_i^m)$ for $m > m_0$ and $h_{m_s} \in W$ for $m_s > m_0$. Since $f_i^{m_s} \upharpoonright K = h_{m_s} \upharpoonright K$ for each $m_s > m_0$, $f_i^{m_s} \in W$ for each $m_s > m_0$. It follows that a sequence $\{f_i^{m_s}\}_{s\in\omega}$ converge to h.

By $C_p(X) \in S_1(\mathcal{S}, \mathcal{A})$, there is a set $\{f_i^{m(i)} : i \in \omega\}$ such that for each i, $f_i^{m(i)} \in \mathcal{S}_i$, and $\{f_i^{m(i)} : i \in \omega\}$ is an element of $\mathcal{A}$.

Consider a set $\{U_i^{m(i)} : i \in \omega\}$.

(a). $U_i^{m(i)} \in \mathcal{U}_i$.

(b). $\{U_i^{m(i)} : i \in \omega\}$ is a cover of X.

Let $x \in X$ and $U =< \mathbf{0}\ , x, \frac{1}{2} >$ be a base neighborhood of $\mathbf{0}$, then there is $f_{i_{j_0}}^{m(i)_{j_0}} \in U$ for some $j_0 \in \omega$. It follows that $x \in U_{i_{j_0}}^{m(i)_{j_0}}$. We thus get $X \models S_1(\Gamma_F, \mathcal{O})$.

(2) $\Leftrightarrow$ (3). By Proposition 13.1.

(3) $\Rightarrow$ (4) is immediate.

(4) $\Rightarrow$ (1). Let $S_n \in \mathcal{S}$ for each $n \in \omega$. We renumber $\{S_n\}_{n\in\omega}$ as $\{S_{i,j}\}_{i,j\in\omega}$. Renumber the rational numbers $\mathbb{Q}$ as $\{q_i : i \in \omega\}$. Fix $i \in \omega$. By

the assumption there exist $f_{i,j} \in S_{i,j}$ such that $\{f_{i,j} : j \in \omega\} \in \mathcal{B}_{q_i}$ where q_i is the constant function to q_i. Then $\{f_{i,j} : i, j \in \omega\} \in \mathcal{A}$.

$\square$

Theorem 13.3. *For a separable metrizable space X, the following statements are equivalent:*

1. $C_p(X) \models S_1(\mathcal{S}, \mathcal{A})$;
2. $X \models S_1(\Gamma_F, \mathcal{O})$;
3. $C_p(X) \models S_1(\Gamma_x, \mathcal{B}_f)$;
4. $C_p(X) \models S_1(\mathcal{S}, \mathcal{B}_f)$.

14. Critical cardinalities

For a collection $\mathcal{J}$ of spaces $C_p(X)$, let $nonC_p(\mathcal{J})$ denote the minimal cardinality for X which $C_p(X)$ is not a member of $\mathcal{J}$.

The critical cardinalities in the Scheepers Diagram [40] are equal to the critical cardinalities of selectors for sequences of countable dense and countable sequentially subsets of $C_p(X)$.

Theorem 14.1. *For a collection $C_p(X)$ of all real-valued continuous functions, defined on Tychonoff spaces X with $iw(X) = \aleph_0$,*

(1) $nonC_p(S_1(\mathcal{D}, \mathcal{S})) = \mathfrak{p}$.

(2) $nonC_p(S_1(\mathcal{S}, \mathcal{S})) = nonC_p(U_{fin}(\mathcal{S}, \mathcal{S})) = \mathfrak{b}$.

(3) $nonC_p(S_{fin}(\mathcal{D}, \mathcal{D})) = nonC_p(S_1(\mathcal{S}, \mathcal{D})) = nonC_p(S_1(\mathcal{S}, \mathcal{A})) = \mathfrak{d}$.

$nonC_p(U_{fin}(\mathcal{S}, \mathcal{D})) = nonC_p(U_{fin}(\mathcal{S}, \mathcal{A})) = nonC_p(S_{fin}(\mathcal{S}, \mathcal{D})) = \mathfrak{d}$.

(4) $nonC_p(S_1(\mathcal{D}, \mathcal{D})) = nonC_p(S_1(\mathcal{A}, \mathcal{A})) = cov(\mathcal{M})$.

We can summarize the relationships between considered notions in next diagrams.

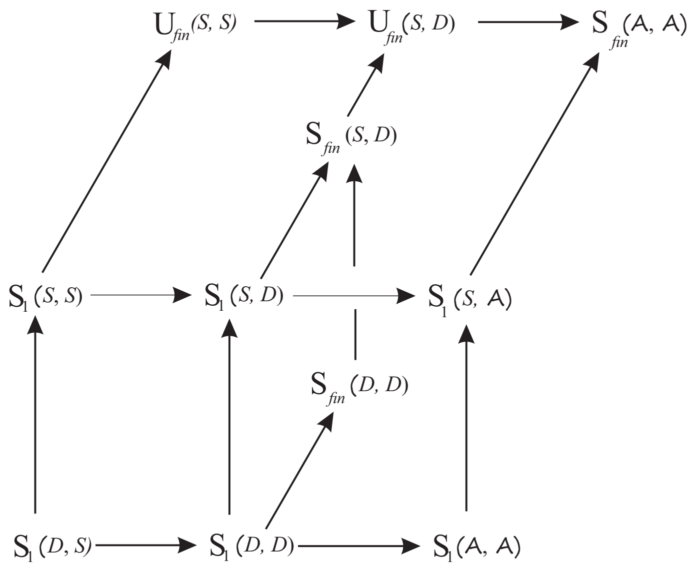

Fig. 2. The Diagram of selectors for sequences of dense sets of $C_p(X)$.

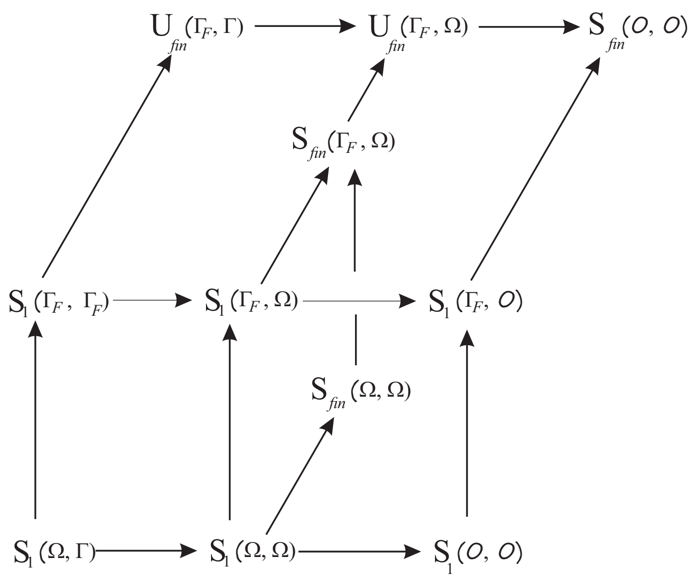

Fig. 3. The Diagram of selection principles for metrizable separable space X corresponding to selectors for sequences of dense sets of $C_p(X)$.

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